Quadratic Finite Volume Element Methods Based on Optimal Stress Points for Solving One-Dimensional Parabolic Problems

Sun Jia-hui [1] , Ma Ming-Juan [2,a] , Pang shi-chun [3] Zhang Yong-po [4]

[1,2,3,4] Aviation university of air force, Changchun, Jilin Province ,china

[a]email mingjuanma@gmail.com

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Abstract. A new Lagrangian quadratic finite volume element method based on optimal stress points was presented for solving one-dimensional parabolic problem with trial and test spaces as the Lagrangian quadratic finite volume element space and the piecewise constant function space respectively. It is proved that the method has optimal order H^1 and L^2 error estimates. The numerical experiment confirms the results of theoretical analysis.

1 Introduction

The finite volume element method (FVEMS) is also called the generalized difference method (GDMS). The method that has a simple format structure, but also has the accuracy of the finite element method, and can keep the quantity of local conservation, it is an effective numerical method for solving partial differential equation of an effective numerical method, it has been widely used in computational fluid mechanics and electromagnetism, etc. At present there are a lot of research results for two-point boundary value problem of finite volume method. In the article , a new Lagrangian quadratic finite volume element method based on optimal stress points was presented for solving one-dimensional parabolic problem. It is proved that the method has optimal order H^1 and L^2 error estimates. The numerical experiment confirms the results of theoretical analysis.

2 Finite volume element format

Consider the mixed problem of one-dimensional parabolic equations on interval I = [a, b] :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial}{dx}(p\dfrac{\partial u}{\partial x}) = f(x,t), & (x,t) \in (a,b)\times(0,T] \\ u(a,t) = 0, \dfrac{\partial u(b,t)}{\partial x} = 0, & t \in (0,T] \\ u(x,0) = u_0(x), & x \in (a,b) \end{cases} \tag{1}$$

Where $p \in C^1(I); p(x) \geq p_{\min} > 0; f \in L^2(I).$ for convenience, recording $u_t = \dfrac{\partial u}{\partial t}, u' = \dfrac{\partial u}{\partial x}.$

The associated weak formulation of problem (1) is:
find $u = u(\cdot,t) \in U := H_E{}^1(I)(0 < t \leq T)$, such that

$$\begin{cases} (u_t, v) + a(u,v) = (f,v), & \forall v \in U, 0 < t \leq T, \\ u(x,0) = u_0(x), & x \in (a,b) \end{cases} \tag{2}$$

where $(\cdot,\cdot)$ express the inner product of $L^2(I)$, $a(u,v) = \int_a^b pu^t v^t dx.$ (3)

So the solution of the equations (2) is called generalized solution of the problem(1)
let a division mesh T_h for $I = [a,b]$,Nodes are $a = x_0 < x_1 < x_2 < \cdots < x_n = b.$

let $h_i = x_i - x_{i-1}, x_{i-1/2} = x_{i-1} + h_i / 2, i = 1, 2, \cdots, n, h = \max_{1\le i\le n} h_i$,

And let the subdivision satisfy regularity conditions $h_i \ge \mu h (i = 1, 2, \cdots n), \mu$ is a positive number. Test space U_h as the Lagrange quadratic finite element space corresponding to T_h. On the unit of $I_i = [x_{i-1}, x_i]$,

$$u_h = u_{i-1}(2\xi - 1) + 4u_{i-1/2}\xi(1-\xi) + u_i(2\xi - 1)\xi = (\xi^2, \xi, 1)\begin{pmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_{i-1} \\ u_{i-1/2} \\ u_i \end{pmatrix} \quad (4)$$

$$u_h' = u_{i-1}(4\xi - 3) + u_{i-1/2}\xi(-8\xi + 4)/h_i + u_i(4\xi - 1)\xi = (\xi, 1)\begin{pmatrix} -4 & 4 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} (u_{i-1/2} - u_{i-1})/h_i \\ (u_i - u_{i-1/2})/h_i \end{pmatrix}. \quad (5)$$

where $\xi = (x - x_{i-1})/h_i$

Then associated to T_h dual mesh T_h^* ,let $e_1 = 1/2 - \sqrt{3}/6, e_2 = 1/2 + \sqrt{3}/6,$ then the two optimal stress points on interval $[x_{j-1}, x_j]$ is $x_{j-e_1} = x_j - e_1 h_j, x_{j-e_2} = x_j - e_2 h_j$, The dual unit node is

$$a = x_0 < x_{e_1} < x_{e_2} < \cdots x_{n-e_2} < x_{n-e_1} < x_n = b.$$

Test function space V_h take as piecewise constant function space corresponding to T_h^* .suppose $\prod_h^*$ is the Interpolation of the projection operator form U_h to V_h. $\prod_h$ is the Interpolation of the projection operator form U to U_h ,then

$$\left|\prod_h u - u\right|_m \le Ch^{2-m}|u|_2, m = 0, 1, \forall u \in U \quad (6)$$

$$\left|\prod_h^* u_h - u_h\right|_0 \le Ch|u_h|_1, \forall u_h \in U_h \quad (7)$$

Suppose J is a natural number, Time step is $\tau = T/J$. define $g^\theta = g(x, t_\theta), t_\theta = \theta\tau$ for every function $g(x,t)$ on $\Omega \times [0,T]$. then define

$\partial_t g^n = (g^n - g^{n-1})/\tau, g^{n-1/2} = (g^n + g^{n-1})/2$ for sequence $\left\{g^n\right\}_{n=0}^{J}$.

Then the Crank-Nicolson Fully discrete finite volume element format that Corresponding to problem (1) is request $u_h^n \in U_h (n = 1, 2, \cdots, J)$, such that

$$\begin{cases} (\partial_t u_h^n, v_h) + a(u_h^{n-1/2}, v_h) = (f^{n-1/2}, v_h), & \forall v_h \in V_h \\ u_h^0 = u_{0h}(x), & x \in (a,b) \end{cases} \quad (8)$$

Where u_{0h} is a discrete approximation of $u_0(x)$, This paper take approximation $\prod_h u_0$ for interpolation or Elliptic projection $R_h u_0$, then there has the following properties:

$$\|u_0 - u_{0h}\|_s \le Ch^{r-s}, s = 0, 1, \quad 2 \le r \le 3. \quad (9)$$

3 Error estimation

Theorem 1 set u is the solution of the problem (1) and u_h^n is the solution of Fully discrete finite volume element format (8), then

1)

$$\left\|u^n-u_h^n\right\|_1 \le C\left\{\left\|u_0-u_{0h}\right\|_1+h^2\left\|u_0\right\|_3+h^2\int_0^{t_n}\left\|u_t\right\|_3 dt+ h^2\left(\int_0^{t_n}\left\|u_t\right\|_3^2 dt\right)^{1/2}+\tau^2\left(\int_0^{t_n}\left\|u_{ttt}\right\|_0 dt\right)^{1/2}\right\}, n=0,1,2,\cdots,J;$$

2)

$$\left\|u^n-u_h^n\right\|_0 \le C\left\{\left\|u_0-u_{0h}\right\|_0+h^3\left\|u_0\right\|_4+h^3\int_0^{t_n}\left\|u_t\right\|_4 dt+\ \tau^2\int_0^{t_n}\left\|u_{ttt}\right\|_0 dt\right\}, n=0,1,2,\cdots,J;$$

3) If the initial value $u_{0h}=\prod_h u_0$ or $R_h u_0$, then there has the following superconvergence estimates:

$$\left\|u^n-\prod_h u^n\right\|_1 \le C(h^3+r^2),$$

$$\left[\frac{1}{r}\sum_{x_0\in N_2}\left|(u^n-u_h^n)'(x_0)\right|^2\right]^{1/2} \le C(h^3+\tau^2),$$

4 Numerical experiments

Consider parabolic problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}-\dfrac{\partial^2 u}{\partial x^2}=0, & 0<x<\dfrac{\pi}{2}, 0<t\le 1, \\ u(0,t)=0, \dfrac{\partial u}{\partial t}u(\dfrac{\pi}{2},t)=0, & 0<t\le 1 \\ u(x,0)=\sin x, & 0<x<\dfrac{\pi}{2}. \end{cases} \quad (10)$$

The exact solution of the problem (10) is $u(x)=e^{-1}\sin x.$

Comparing to the two methods show that the new method has a better convergence.

Table 1 Convergence orders of the method (8) for the problem (10)

n	$\left\|u^n-u_h^n\right\|_0$	Convergence order	$\left\|u^n-u_h^n\right\|_1$	Convergence order	$\left\|u_h^n-\prod_h u^n\right\|_1$	Convergence order
8	2.4610×10^{-5}		7.8417×10^{-4}		1.0022×10^{-5}	
16	2.9638×10^{-6}	3.0537	1.9381×10^{-4}	2.0165	1.0839×10^{-6}	3.2089
32	3.6684×10^{-7}	3.0143	4.8315×10^{-5}	2.0041	1.2952×10^{-7}	3.0649
64	4.5739×10^{-8}	3.0036	1.2070×10^{-5}	2.0010	1.5997×10^{-8}	3.0173
128	5.7138×10^{-9}	3.0009	3.0170×10^{-6}	2.0003	1.9935×10^{-9}	3.0044

Table 2 Convergence orders of usual quadratic FVEM for the problem (11)

n	$\left\|u^n-u_h^n\right\|_0$	Convergence order	$\left\|u^n-u_h^n\right\|_1$	Convergence order	$\left\|u_h^n-\prod_h u^n\right\|_1$	Convergence order
8	1.0234×10^{-4}		7.9049×10^{-4}		1.0039×10^{-4}	
16	2.6619×10^{-5}	1.9248	1.9561×10^{-4}	2.0147	2.6509×10^{-5}	1.9211
32	6.7229×10^{-6}	1.9853	4.8780×10^{-5}	2.0037	6.7162×10^{-6}	1.9808
64	1.6850×10^{-6}	1.9963	1.2187×10^{-5}	2.0009	1.6846×10^{-6}	1.9952
128	4.2153×10^{-7}	1.9991	3.0463×10^{-6}	2.0002	4.2150×10^{-7}	1.9988

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