

Quantum Teleportation of a Three-qubit State using a Four-qubit Entangled State

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Keywords: Quantum Teleportation, Three-qubit State, Four-qubit Entangled State

Abstract. Recently Liu et al.(Int. J. Theor. Phys. **53**:4079, 2014) had shown that a special form of three-qubit entangled state $|\phi\rangle = \alpha|000\rangle + \beta|001\rangle + \gamma|111\rangle + \delta|110\rangle$ can be teleported by using a five-qubit cluster state. However, we demonstrate that five-qubit cluster state and five-qubit von-Neumann projective measurements are not necessary to complete the protocol. The special three-qubit state $|\phi\rangle$ can be teleported via a four-qubit entangled state, by introducing one ancillary qubit and one controlled-not operation. Physical realization of the proposed four-qubit entangled state and the generalization to teleport the multi-qubit state is also presented.

Introduction

Quantum teleportation is one of the most astonishing features of quantum mechanics. An unknown quantum state can be teleported from one site to another via previously shared entanglement assisted by classical communications and local operations. In 1993, Bennett et al. [1] proposed the first protocol of quantum teleportation of an arbitrary single-qubit state using a maximally entangled two-qubit state. Four years later, this protocol was experimentally demonstrated [2]. Thereafter, teleportation of an arbitrary single-qubit state was proposed using tripartite GHZ state [3], four-partite GHZ state [4], SLOCC equivalent W-class state [5], cluster state [6], etc. Teleportation of an arbitrary two-qubit state was proposed using tensor product of two Bell states [7], tensor product of two orthogonal states [8], genuinely entangled five-qubit state [9], five-qubit cluster state [10], six-qubit genuinely entangled state [11], etc.

Recently, Liu et al. [12] had shown that a special form of three-qubit entangled state $|\phi\rangle = \alpha|000\rangle + \beta|001\rangle + \gamma|111\rangle + \delta|110\rangle$ can be teleported by using a five-qubit cluster state based on five-qubit von-Neumann projective measurements and local unitary operations. However, we demonstrate that five-qubit cluster state and five-qubit von-Neumann projective measurements are not necessary to complete the protocol. The special three-qubit state $|\phi\rangle$ can be teleported via a four-qubit entangled state, by introducing one ancillary qubit and one controlled-not (CNOT) operation. The proposed four-qubit entangled state can be physically realized by a pair of Bell states and two single-qubit states. The generalization of the protocol to teleport a multi-qubit state is also presented.

Quantum Teleportation of a Three-qubit State

In [12], a special form of three-qubit state is given by

$$|\phi\rangle_{abc} = (\alpha|000\rangle + \beta|001\rangle + \gamma|111\rangle + \delta|110\rangle)_{abc} \quad (1)$$

where $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$.

Our teleportation scheme can be described as follows. Suppose Alice has an unknown three-qubit state $|\phi\rangle_{abc}$. She wants to send this state to a distant receiver Bob. Alice and Bob share a four-qubit entangled state

$$|C\rangle_{1234} = \frac{1}{2}(|0000\rangle + |0111\rangle + |1101\rangle + |1010\rangle)_{1234} \quad (2)$$

with the qubits a,b,c,3,4 belong to Alice, qubits 1,2 belong to Bob, respectively. The joint state of the three-qubit state and the quantum channel is given by

$$|\Omega\rangle_{abc1234} = |\phi\rangle_{abc} \otimes |C'\rangle_{1234}. \quad (3)$$

Alice first applies a CNOT operation on qubits a and b , with qubit b as control and qubit a as target, which will result in $|0\rangle_a \otimes (\alpha|00\rangle + \beta|01\rangle + \gamma|11\rangle + \delta|10\rangle)_{bc}$. Now, we note that all the information of the state $(\alpha|000\rangle + \beta|001\rangle + \gamma|111\rangle + \delta|110\rangle)_{abc}$ has been transferred into qubit b, c with the state $(\alpha|00\rangle + \beta|01\rangle + \gamma|11\rangle + \delta|10\rangle)_{bc}$. So we only need to consider the state of qubits $b, c, 1, 2, 3, 4$ which is

$$\begin{aligned} & (\alpha|00\rangle + \beta|01\rangle + \gamma|11\rangle + \delta|10\rangle)_{bc} \otimes |C'\rangle_{1234} \\ &= \frac{1}{4}|\varphi^1\rangle_{bc34}(\alpha|00\rangle + \beta|01\rangle + \gamma|11\rangle + \delta|10\rangle)_{12} + \frac{1}{4}|\varphi^2\rangle_{bc34}(\alpha|00\rangle - \beta|01\rangle - \gamma|11\rangle + \delta|10\rangle)_{12} \\ &+ \frac{1}{4}|\varphi^3\rangle_{bc34}(\alpha|00\rangle - \beta|01\rangle + \gamma|11\rangle - \delta|10\rangle)_{12} + \frac{1}{4}|\varphi^4\rangle_{bc34}(\alpha|00\rangle + \beta|01\rangle - \gamma|11\rangle - \delta|10\rangle)_{12} \\ &+ \frac{1}{4}|\varphi^5\rangle_{bc34}(\alpha|01\rangle + \beta|00\rangle + \gamma|10\rangle + \delta|11\rangle)_{12} + \frac{1}{4}|\varphi^6\rangle_{bc34}(\alpha|01\rangle - \beta|00\rangle + \gamma|10\rangle - \delta|11\rangle)_{12} \\ &+ \frac{1}{4}|\varphi^7\rangle_{bc34}(\alpha|01\rangle - \beta|00\rangle - \gamma|10\rangle + \delta|11\rangle)_{12} + \frac{1}{4}|\varphi^8\rangle_{bc34}(\alpha|01\rangle + \beta|00\rangle - \gamma|10\rangle - \delta|11\rangle)_{12} \\ &+ \frac{1}{4}|\varphi^9\rangle_{bc34}(\alpha|11\rangle + \beta|10\rangle + \gamma|00\rangle + \delta|01\rangle)_{12} + \frac{1}{4}|\varphi^{10}\rangle_{bc34}(\alpha|11\rangle - \beta|10\rangle - \gamma|00\rangle + \delta|01\rangle)_{12} \\ &+ \frac{1}{4}|\varphi^{11}\rangle_{bc34}(\alpha|11\rangle + \beta|10\rangle - \gamma|00\rangle - \delta|01\rangle)_{12} + \frac{1}{4}|\varphi^{12}\rangle_{bc34}(\alpha|11\rangle - \beta|10\rangle + \gamma|00\rangle - \delta|01\rangle)_{12} \\ &+ \frac{1}{4}|\varphi^{13}\rangle_{bc34}(\alpha|10\rangle + \beta|11\rangle + \gamma|01\rangle + \delta|00\rangle)_{12} + \frac{1}{4}|\varphi^{14}\rangle_{bc34}(\alpha|10\rangle - \beta|11\rangle + \gamma|01\rangle - \delta|00\rangle)_{12} \\ &+ \frac{1}{4}|\varphi^{15}\rangle_{bc34}(\alpha|10\rangle + \beta|11\rangle - \gamma|01\rangle - \delta|00\rangle)_{12} + \frac{1}{4}|\varphi^{16}\rangle_{bc34}(\alpha|10\rangle - \beta|11\rangle - \gamma|01\rangle + \delta|00\rangle)_{12} \end{aligned} \quad (4)$$

where $|\varphi^i\rangle_{bc34}$ ($i = 1, 2, 3, \dots, 16$) are mutually orthogonal states in Alice's possession given by

$$\begin{aligned} |\varphi^1\rangle &= \frac{1}{2}(|0000\rangle + |0111\rangle + |1101\rangle + |1010\rangle), |\varphi^2\rangle = \frac{1}{2}(|0000\rangle - |0111\rangle - |1101\rangle + |1010\rangle), \\ |\varphi^3\rangle &= \frac{1}{2}(|0000\rangle - |0111\rangle + |1101\rangle - |1010\rangle), |\varphi^4\rangle = \frac{1}{2}(|0000\rangle + |0111\rangle - |1101\rangle - |1010\rangle), \\ |\varphi^5\rangle &= \frac{1}{2}(|0011\rangle + |0100\rangle + |1001\rangle + |1110\rangle), |\varphi^6\rangle = \frac{1}{2}(|0011\rangle - |0100\rangle - |1001\rangle + |1110\rangle), \\ |\varphi^7\rangle &= \frac{1}{2}(|0011\rangle - |0100\rangle + |1001\rangle - |1110\rangle), |\varphi^8\rangle = \frac{1}{2}(|0011\rangle + |0100\rangle - |1001\rangle - |1110\rangle), \\ |\varphi^9\rangle &= \frac{1}{2}(|0001\rangle + |1100\rangle + |0110\rangle + |1011\rangle), |\varphi^{10}\rangle = \frac{1}{2}(|0001\rangle - |1100\rangle - |0110\rangle + |1011\rangle), \\ |\varphi^{11}\rangle &= \frac{1}{2}(|0001\rangle - |1100\rangle + |0110\rangle - |1011\rangle), |\varphi^{12}\rangle = \frac{1}{2}(|0001\rangle + |1100\rangle - |0110\rangle - |1011\rangle), \\ |\varphi^{13}\rangle &= \frac{1}{2}(|0010\rangle + |1000\rangle + |0101\rangle + |1111\rangle), |\varphi^{14}\rangle = \frac{1}{2}(|0010\rangle - |1000\rangle - |0101\rangle + |1111\rangle), \\ |\varphi^{15}\rangle &= \frac{1}{2}(|0010\rangle - |1000\rangle + |0101\rangle - |1111\rangle), |\varphi^{16}\rangle = \frac{1}{2}(|0010\rangle + |1000\rangle - |0101\rangle - |1111\rangle). \end{aligned} \quad (5)$$

After the measurements in the basis of $\{|\varphi^i\rangle_{bc34} (i = 1, 2, 3, \dots, 16)\}$ on her qubits $b, c, 3, 4$, Alice communicates her measurement results via four classical bits to Bob. In order to recover the original three-qubit state, Bob first applies appropriate Pauli rotations on qubits (1, 2). Then Bob introduces one ancillary qubit A in the initial state $|0\rangle_A$ and performs CNOT operation on qubits 1 and A , with qubit 1 as control and qubit A as target. Alice's measurement result and Bob's corresponding operations are listed in Table 1.

Physical Realization and Generalization

The proposed four-qubit entangled state can be physically realized as follows. We start with two photons in the Bell state $|\psi^+\rangle_{12} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{12}$. We need to prepare another photon in the state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_3$. One can combine both these states and get a W class of states as follows:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{12} \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_3 = \frac{1}{2}(|000\rangle + |010\rangle + |101\rangle + |111\rangle)_{132}. \quad (6)$$

We now take another photon in the state $|0\rangle_A$. The four-qubit entangled state can be obtained by applying a CNOT operation on qubits $A, 1, 3$, with qubits $(1, 3)$ as control and qubit A as target:

$$\frac{1}{2}|0\rangle_A(|000\rangle + |010\rangle + |101\rangle + |111\rangle)_{132} \implies \frac{1}{2}(|0000\rangle + |1010\rangle + |1101\rangle + |0111\rangle). \quad (7)$$

By introducing N ancillary qubits and performing N CNOT operations, our scheme can be easily generalized to teleport a $(N + 2)$ -qubit entangled state $|\xi\rangle$ given by

$$\begin{aligned} N = 1, & (\alpha|000\rangle + \beta|001\rangle + \gamma|111\rangle + \delta|110\rangle)_{1bc}, \\ N = 2, & (\alpha|0000\rangle + \beta|0001\rangle + \gamma|1111\rangle + \delta|1110\rangle)_{12bc}, \\ N = 3, & (\alpha|00000\rangle + \beta|00001\rangle + \gamma|11111\rangle + \delta|11110\rangle)_{123bc}, \\ N = n, & (\alpha|\underbrace{0\dots0}_n00\rangle + \beta|\underbrace{0\dots0}_n01\rangle + \gamma|\underbrace{1\dots1}_n11\rangle + \delta|\underbrace{1\dots1}_n10\rangle)_{12\dots nbc}. \end{aligned} \quad (8)$$

The generalized scheme can be described briefly as follows.

To teleport a special $(N + 2)$ -qubit state $|\xi\rangle$, which is the form of Eq. (8), Alice first applies N CNOT operations on qubits $1, 2, \dots, N, b$ with qubit b as control and qubits $1, 2, \dots, N$ as target, which will result in $\otimes_{k=1}^N |0\rangle_k \otimes (\alpha|00\rangle + \beta|01\rangle + \gamma|11\rangle + \delta|10\rangle)_{bc}$. So all the information of $|\xi\rangle$ has been transferred into qubit b, c with the state $\alpha|00\rangle + \beta|01\rangle + \gamma|11\rangle + \delta|10\rangle$. We only need to consider the joint state $(\alpha|00\rangle + \beta|01\rangle + \gamma|11\rangle + \delta|10\rangle)_{bc} \otimes |C\rangle_{defg}$, which is showed in Eq. (4) and (5). After the measurements in the basis of $\{|\varphi^i\rangle (i = 1, 2, 3, \dots, 16)\}$ on her qubits b, c, f, g , Alice communicates her measurement results via four classical bits to Bob. In order to recover the original $(N + 2)$ -qubit state, Bob first applies appropriate Pauli rotations on qubits (d, e) . Then Bob introduces N ancillary qubits $1, 2, \dots, N$ in the initial state $\otimes_{k=1}^N |0\rangle_k$ and performs N CNOT operations on qubits d and $1, 2, \dots, N$, with qubit d as control and qubits $1, 2, \dots, N$ as target respectively. Alice's measurement result and Bob's corresponding operations can be referred to Table 1.

Alice's result	Bob's state	Bob's operation
$ \varphi^1\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 00\rangle + \beta 01\rangle + \gamma 11\rangle + \delta 10\rangle)_{12}$	$CNOT_{A1}(I_1 \otimes I_2)$
$ \varphi^2\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 00\rangle - \beta 01\rangle - \gamma 11\rangle + \delta 10\rangle)_{12}$	$CNOT_{A1}(I_1 \otimes \delta_2^z)$
$ \varphi^3\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 00\rangle - \beta 01\rangle + \gamma 11\rangle - \delta 10\rangle)_{12}$	$CNOT_{A1}(\delta_1^z \otimes \delta_2^z)$
$ \varphi^4\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 00\rangle + \beta 01\rangle - \gamma 11\rangle - \delta 10\rangle)_{12}$	$CNOT_{A1}(\delta_1^z \otimes I_2)$
$ \varphi^5\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 01\rangle + \beta 00\rangle + \gamma 10\rangle + \delta 11\rangle)_{12}$	$CNOT_{A1}(I_1 \otimes \delta_2^x)$
$ \varphi^6\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 01\rangle - \beta 00\rangle + \gamma 10\rangle - \delta 11\rangle)_{12}$	$CNOT_{A1}(\delta_1^z \otimes \delta_2^z \delta_2^x)$
$ \varphi^7\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 01\rangle - \beta 00\rangle - \gamma 10\rangle + \delta 11\rangle)_{12}$	$CNOT_{A1}(I_1 \otimes \delta_2^z \delta_2^x)$
$ \varphi^8\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 01\rangle + \beta 00\rangle - \gamma 10\rangle - \delta 11\rangle)_{12}$	$CNOT_{A1}(\delta_1^z \otimes \delta_2^x)$
$ \varphi^9\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 11\rangle + \beta 10\rangle + \gamma 00\rangle + \delta 01\rangle)_{12}$	$CNOT_{A1}(\delta_1^x \otimes \delta_2^x)$
$ \varphi^{10}\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 11\rangle - \beta 10\rangle - \gamma 00\rangle + \delta 01\rangle)_{12}$	$CNOT_{A1}(\delta_1^x \otimes \delta_2^z \delta_2^x)$
$ \varphi^{11}\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 11\rangle + \beta 10\rangle - \gamma 00\rangle - \delta 01\rangle)_{12}$	$CNOT_{A1}(\delta_1^x \delta_1^x \otimes \delta_2^x)$
$ \varphi^{12}\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 11\rangle - \beta 10\rangle + \gamma 00\rangle - \delta 01\rangle)_{12}$	$CNOT_{A1}(\delta_1^z \delta_1^x \otimes \delta_2^z \delta_2^x)$
$ \varphi^{13}\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 10\rangle + \beta 11\rangle + \gamma 01\rangle + \delta 00\rangle)_{12}$	$CNOT_{A1}(\delta_1^x \otimes I_2)$
$ \varphi^{14}\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 10\rangle - \beta 11\rangle + \gamma 01\rangle - \delta 00\rangle)_{12}$	$CNOT_{A1}(\delta_1^z \delta_1^x \otimes \delta_2^z)$
$ \varphi^{15}\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 10\rangle + \beta 11\rangle - \gamma 01\rangle - \delta 00\rangle)_{12}$	$CNOT_{A1}(\delta_1^z \delta_1^x \otimes I_2)$
$ \varphi^{16}\rangle_{bc34}$	$ 0\rangle_A \otimes (\alpha 10\rangle - \beta 11\rangle - \gamma 01\rangle + \delta 00\rangle)_{12}$	$CNOT_{A1}(\delta_1^x \otimes \delta_2^z)$

Table 1. Strategy for recovering the three-qubit state. $CNOT_{A1}$ represents the controlled-NOT operation on qubits 1 and A, with qubit 1 as control and qubit A as target. In δ_k^z and δ_k^x , the subindex k indicates on which qubit the Pauli matrices δ^z and δ^x should operate.

Conclusion

In summary, we have demonstrated that a restricted class of three-qubit state can be faithfully and deterministically teleported via a four-qubit entangled state by introducing one ancillary qubit and performing one CNOT operation. Only four-qubit von-Neumann projective measurements, one CNOT operation and local unitary operations are needed, which means that five-qubit cluster state

and five-qubit von-Neumann projective measurements are not necessary to complete the protocol. By introducing N ancillary qubits and performing N CNOT operations, our scheme can be easily generalized to teleport a special multi-qubit state.

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