Some properties and applicatons of (inverse)L_{n-1}-matrices

Yang Chuansheng[1, a *], Wang Chao[1,b]

[1]School of Mathematics, Zhejiang Ocean University

Zhoushan, Zhejiang Province, P. R. China, 316011

[a]cshyang@zjou.edu.cn, [b]wangchao1321654@163.com

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Abstract. In this paper we study some combinatorial properties and inequalities of some classes of Z-matrices. These matrices arise in many problems in the mathematical and physical sciences. We show that all (inverse) L_{n-1}-matrices are irreducible and some eigenvalue inequalities of (inverse) L_{n-1}-matrices.

Introduction and Notation

Throughout we deal with $n\times n$ *Z*-matrices, i.e. real matrices whose off-diagonal entries are nonpositive. These matrices arise in many problems in the mathematical and physical sciences. Some of the best-known subclasses of *Z*-matrices are the class of *M*-matrices (introduced by Ostrowski), the class of L_{n-1}-matrices (introduced by Ky Fan and G. A. Johnson), and the class of F_0-matrices (introduced by G. A. Johnson). Especially for *M*-matrices, a large number of properties and characterizations exist. However, the other classes of *Z*-matrices are also of great interest. In this paper we study some combinatorial properties and inequalities of L_{n-1}-matrices and inverse L_{n-1}-matrices. Firstly we introduce some definitions.

In 1992 Fiedler and Markham [1] introduced the following classification of Z-matrices:

Definition 1 Let L_s (for *s*=0,1,…, *n*) denote the class of matrices consisting of real $n\times n$ matrices which have the form

$$A = tI - B\text{, where } B \ge 0 \text{ and } r_s(B) \le t < r_{s+1}(B), \tag{1}$$

here

$$r_s(B) = \max\{r(\tilde{B}) : \tilde{B} \text{ is an } s\times s \text{ principal submatrix of } B\},$$

and we set $r_0(B) = -\infty$ and $r_{n+1}(B) = \infty$.

The scalar *t* and the matrix *B* in (1) are not unique, but every Z-matrix belongs to exactly one set L_s . Moreover, none of the class L_s is void. However, if one considers a fixed matrix *B* in (1), some of the class L_s can be the same, since we have in general only that

$$r_1(B) \le r_2(B) \le \mathbf{L} \le r_n(B)$$

The class L_n is just the class of $n\times n$ (singular and nonsingular) *M*-matrices. The class L_{n-1} is introduced by G. A. Johnson [2], and this class contains the *N*-matrices defined by Ky Fan [3]. Moreover, the class of $n\times n$ F_0-matrices introduced by Johnson [2] is just L_{n-2} . Here we should mention that the classification of Z-matrices given above inherits the dimension of the matrices one considers. If we deal with $n\times n$ matrices, we have *n*+1classes of Z-matrices, each consisting of matrices of the same dimension.

As proved in [1], for each *s* with $1\le s\le n-1$, the class L_s is equal to the class of *Z*-matrices for which all principal submatrices of order *s* are *M*-matrices, but there exists a principal submatrix of order *s*+1 which is not an *M*-matrix. Additional properties of some of these classes are given in [4, 5, 6].

On the other hand, there has been interest in inverse *M*-matrices, i.e., any nonsingular matrix $B\ge 0$ whose inverse is an M-matrix. A survey of this topic is given by C. R. Johnson [7]. Thus, it is natural to determine classes of matrices which are inverse Z-matrices. A first step was taken by G. A. Johnson [8], who proved that a matrix of negative type D is an inverse L_{n-1}-matrix. Later, Chen [9] studied necessary conditions for a matrix to be an inverse F_0-matrix; some new results on Z-matrices are in [10].

At last, for two $m\times n$ matrices $A=(a_{ij})$ and $B=(b_{ij})$, the Hadamard product of $A\circ B$ is defined and denoted by $A\circ B=(a_{ij}b_{ij})$.

In this paper, we show that all (inverse) L_{n-1}-matrices are irreducible and some eigenvalue inequalities of (inverse) L_{n-1}-matrices.

Result and proof

In this section, we firstly introduce some combinatorial properties of L_{n-1}-matrices and inverse L_{n-1}-matrices.

Theorem 1: All (inverse) L_{n-1}-matrices are irreducible.

Proof: We firstly show that all L_{n-1}-marices are irreducible.

Assume that a L_{n-1}-matrix A is reducible, then there exists a permutation matrix P such that

$$P^T AP=\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}=tI-\begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}, \tag{2}$$

where A_1 and A_2 are $s\times s\,(1\le s<n)$ and $(n-s)\times(n-s)$ matrices respectively, $B=\begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}\ge 0$.

Since $A\in L_{n-1}$, then $P^T AP\in L_{n-1}$ and $r_{n-1}(B)\le t<r(B)$. But $r(B)=\max\{r(B_1), r(B_3)\}$, then $r(B)\le r_{n-1}(B)$, this is a contradiction, so all L_{n-1}-marices are irreducible.

At lastly, we prove that all inverse L_{n-1}-marices are irreducible.

Assume that an inverse L_{n-1}-matrix A is reducible, then there exists a permutation matrix P such that

$$P^T AP=\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}=\tilde{A}, \tag{3}$$

then $\tilde{A}$ is an inverse L_{n-1}-matrix and $\tilde{A}^{-1}$ is a L_{n-1}-matrix, but

$$\tilde{A}^{-1}=\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}^{-1}=\begin{pmatrix} A_1^{-1} & -A_1^{-1}A_2A_3^{-1} \\ 0 & A_3^{-1} \end{pmatrix}$$

is reducible. We know that all L_{n-1}-marices are irreducible, this is a contradiction, so all inverse L_{n-1}-marices are irreducible. This completes the proof of this theorem.

For establishing some results on eigenvalues of (inverse) L_{n-1}-matrices, we need the following lemma.

Lemma 2[11]: Assume that an $n\times n$ $A\ge 0$, then any real eigenvalue l of A different from $r(A)$ satisfies the inequality

$$l\le r_{\lfloor n/2\rfloor}(A), \tag{4}$$

If A is positive, then the inequality (4) is strict.

Lemma 3(Gersgorin's theorem): For any $A=(a_{ij})\in C^{n\times n}$ and any eigenvalue $l \in s(A)$, there is a positive k in N={1,2,…,n} such that

$$|l-a_{kk}|\le \sum_{j\in N|\{k\}} |a_{kj}|, \tag{5}$$

where $s(A)=\{l\in C : \det(lI-A)=0\}$.

Theorem 4: Let A be an inverse L_{n-1}-matrix, then A has exactly one negative eigenvalue and det A<0.

Proof: Since $A^{-1}\in L_{n-1}$, then $A^{-1}=tI-B\ (B\ge 0, r_{n-1}(B)\le t<r(B))$, so A^{-1} has a negative eigenvalue $t-r(B)$. According to Lemma 1, we know that A^{-1} has no other negative eigenvalues, then A has a negative eigenvalue $(t-r(B))^{-1}$, so det A<0. This completes the proof.

Theorem 5: Let $A=(a_{ij})_{n\times n}$ be an inverse L_{n-1}-matrix, $A^{-1}=(\overline{a_{ij}})_{n\times n}$, then

$$q(A\circ A^{-1})>\left(|a_{ii}|-\sum_{j\in N\backslash\{i\}}|a_{ij}|\frac{|a_{ji}|+\sum_{k\in N\backslash\{i,j\}}|a_{jk}|d_k}{|a_{jj}|}\right)|\overline{a_{ii}}|, \tag{6}$$

where $d_k=\frac{\sum_{j\in N\backslash\{k\}}|a_{kj}|}{|a_{kk}|}$ and $q(A)=\min\{|l| :\ l\in s(A)\}$.

Proof: Let $l\in s(A\circ A^{-1})$ and $|l|=q(A\circ A^{-1})$. According to lemma 3, then there exists a $i\in N$ such that

$$|l-a_{ii}\overline{a_{ii}}|\le \sum_{j\in N\backslash\{i\}}|a_{ij}\overline{a_{ij}}|,$$

then

$$|l|\ge|a_{ii}\overline{a_{ii}}|-\sum_{j\in N\backslash\{i\}}|a_{ij}\overline{a_{ij}}|$$

$$\ge|a_{ii}\overline{a_{ii}}|-\sum_{j\in N\backslash\{i\}}|a_{ij}|\sum_{j\in N\backslash\{i\}}|a_{ij}|\frac{|a_{ji}|+\sum_{k\in N\backslash\{i,j\}}|a_{jk}|d_k}{|a_{jj}|}|\overline{a_{ii}}|$$

$$=\left(|a_{ii}|-\sum_{j\in N\backslash\{i\}}|a_{ij}|\frac{|a_{ji}|+\sum_{k\in N\backslash\{i,j\}}|a_{jk}|d_k}{|a_{jj}|}\right)|\overline{a_{ii}}|.$$

This completes the proof of the theorem.

Conclusions

In this paper some combinatorial results and inequalities of some classes of Z-matrices and inverse Z-matrices are given. We show that all (inverse) L_{n-1}- matrices are irreducible and some eigenvalue inequalities of (inverse) L_{n-1}- matrices. These results have wide applications in the mathematical and physical sciences.

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