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A Lifetime Distribution Based on a Transformation of a Two-Sided Power Variate

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We introduce a new generalization of Weibull distribution by making use of a transformation of the standard two- sided power distributed random variable. Weibull and the exponentiated Weibull distributions are sub-models of this new distribution. We show that this newly defined distribution is in fact a mixture of the truncated forms of Weibull and the exponentiated Weibull distributions. The new distribution has two shape parameters that make it more flexible for modeling data than Weibull and exponentiated Weibull distributions. We study its properties, consider the maximum likelihood estimation procedure and apply it on some real data sets from reliability.

Keywords: generalized Weibull distribution; maximum likelihood estimation; mixture distribution; reliability; two-sided power distribution

2000 Mathematics Subject Classification: 60E05, 62H12

1. Introduction

The standard two-sided power distribution, denoted by TSP, is introduced by [31] and defined by the following probability density function (pdf)

$$f_{TSP}(x;\alpha,\beta) = \begin{cases} \alpha\left(\dfrac{x}{\beta}\right)^{\alpha-1}, & 0 < x \leq \beta \\ \alpha\left(\dfrac{1-x}{1-\beta}\right)^{\alpha-1}, & \beta \leq x < 1, \end{cases}$$

where $0 < \beta < 1$ and $\alpha > 0$. The distribution is denoted by $TSP(\alpha,\beta)$. The parameter β is the *reflection parameter* and α is the *shape parameter*. The TSP distribution is one of the beta-like distributions. It is defined on a bounded support. The parameters in the distribution determine the shapes of the distribution and they are similar to those of the beta distribution. For example, for $0 < \beta < 1$ and $\alpha > 1$, the distribution is unimodal; for $0 < \beta < 1$ and $0 < \alpha < 1$, the distribution

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is U-shaped with mode 0 or 1; for $\alpha = 1$, the distribution is uniform on (0,1); for $\alpha = 2$, the distribution is triangular. The TSP distribution is clearly more flexible than the power function distribution which is obtained for the case $\beta = 1$. Compared to the beta distribution the TSP has the advantage of having its cumulative distribution function (cdf) explicitly:

$$F_{TSP}(x;\alpha,\beta) = \begin{cases} \beta\left(\dfrac{x}{\beta}\right)^{\alpha}, & 0 < x \leq \beta \\ 1-(1-\beta)\left(\dfrac{1-x}{1-\beta}\right)^{\alpha}, & \beta \leq x < 1 \end{cases}$$

The TSP distribution is useful for modeling financial data where peaked cases are more frequently observed (see [31], [32], [8], [27]). Kurtosis properties of the distribution were studied by [9]. It can be easily seen that the TSP distribution is an open distribution to generalizations and some of them were defined and studied by several authors e. g. [33], [19], [26], [3], [30]. Similar to the way of the definition of the TSP, [34] defined and studied the two-sided generalized Topp and Leone distribution. Recently, [7] considered the log transform of the distribution to obtain a generalized exponential distribution, and studied the new distribution in detail.

On the other hand, the ordinary Weibull distribution with two parameters, denoted by $W(\gamma,\theta)$, has the pdf

$$f_W(x;\gamma,\theta) = \frac{\gamma}{\theta^{\gamma}}x^{\gamma-1}e^{-(x/\theta)^{\gamma}}, x > 0,$$

where $\gamma > 0$ and $\theta > 0$ are the shape and scale parameters, respectively. For $\gamma \leq 1$, the distribution becomes J-shaped; for $\gamma = 1$, the distribution is reduced to exponential distribution and for $\gamma > 1$, the distribution becomes bell-shaped. Weibull distribution is known to be one of the most commonly used distributions in reliability and in life testing studies (see e.g. [4], [17]). It is flexible in the sense that it has an increasing, decreasing or constant failure rate according to its shape parameter.

However, other hazard rate shapes, that is, non-monotone hazard rates are also common in practice and Weibull distribution is insufficient at this point. To a remedy, several generalizations of the ordinary Weibull distribution have been introduced in the literature. One approach to increase the flexibility and allow for also non-monotone hazard rate modeling is made by adding an additional shape parameter appropriately to the survival function. For example, the extended Weibull ([13]) and the exponentiated Weibull ([16]). Another approach can use transformed variates and this procedure is usually resulted in models that can have different shapes from the untransformed one. For example, logarithmic distributions obtained from log-transformations are useful in statistics (see e.g. Chap. 12 in [14] and [25]). As another example, power transforms of random variables can be given (see p. 148 in [6] and p. 228 in [14]). A good review on Weibull and its extensions is given in [10].

The exponentiated Weibull distribution, denoted by $EW(\alpha,\gamma,\theta)$, which was introduced by [16] and has the pdf

$$f_{EW}(x;\alpha,\gamma,\theta) = \frac{\alpha\gamma x^{\gamma-1}}{\theta^{\gamma}}\exp\left[-\left(\frac{x}{\theta}\right)^{\gamma}\right]\left\{1-\exp\left[-\left(\frac{x}{\theta}\right)^{\gamma}\right]\right\}^{\alpha-1}, \quad x > 0$$

where $\alpha > 0$ and $\gamma > 0$ are the shape parameters and $\theta > 0$ is the scale parameter. The authors discussed some properties of this distribution and derived maximum likelihood estimators. [22] discussed some statistical properties such as mode, moments, failure rate and mean residual life of the *EW* distribution. Recently, Bayesian estimation this distribution under type II progressive censoring was considered by [5]. A survey paper on this distribution was given by [21].

Since the *TSP* distribution is well-known for its usefulness in modeling data with high kurtosis, the aim of this paper is to propose a useful extension of Weibull distribution like the *TSP* in addition to those existing ones. In order to obtain the new extension, we make the log power transformation of the *TSP*. The new distribution has four parameters and it generalizes the exponential, Weibull, generalized exponential introduced by [2], *EW*, Burr type *X* and two-sided generalized exponential distribution (*TSGE*) introduced by [7].

The rest of the paper is organized as follows. In Section 2, we define the new distribution and study its density shapes in detail. We derive the moments, hazard function and Rnyi entropy of the distribution in Sections 3, 4 and 5 respectively. Section 6 is devoted to maximum likelihood estimation procedure and a simulation study conducted to see the performance of the proposed estimators. The two real data sets are analyzed in Section 7. Finally we end the paper with some concluding remarks.

2. Definition of the New Generalized Weibull Distribution

2.1. *Definition*

Since the *TSP* distribution generalizes the uniform distribution we naturally use this generalization in the derivation of the extension of the ordinary Weibull distribution. Consequently, we consider making the transformation $X = \theta(-\log Y)^{1/\gamma}$, where $Y \sim TSP(\alpha,\beta)$. We then obtain the cdf of X as

$$F(x;\alpha,\gamma,\theta,\beta) = \begin{cases} (1-\beta)^{1-\alpha}\left\{1-\exp\left[-\left(\frac{x}{\theta}\right)^{\gamma}\right]\right\}^{\alpha}, & 0 < x \le \eta \\ 1-\beta^{1-\alpha}\exp\left[-\alpha\left(\frac{x}{\theta}\right)^{\gamma}\right], & \eta \le x < \infty, \end{cases}$$

and the pdf is given by

$$f(x;\alpha,\gamma,\theta,\beta) = \begin{cases} \frac{\alpha\gamma(1-\beta)^{1-\alpha}}{\theta^{\gamma}}x^{\gamma-1}\exp\left[-\left(\frac{x}{\theta}\right)^{\gamma}\right]\left\{1-\exp\left[-\left(\frac{x}{\theta}\right)^{\gamma}\right]\right\}^{\alpha-1}, & 0 < x \le \eta \\ \frac{\alpha\gamma\beta^{1-\alpha}}{\theta^{\gamma}}x^{\gamma-1}\exp\left[-\alpha\left(\frac{x}{\theta}\right)^{\gamma}\right], & \eta \le x < \infty, \end{cases} \tag{2.1}$$

where $\eta = \theta(-\log\beta)^{1/\gamma}$, $\alpha,\gamma,\theta > 0$ and $0 < \beta < 1$. When $\alpha = 1$, the pdf in (2.1) is reduced to the ordinary Weibull distribution. Therefore, the distribution of X is a generalization of Weibull distribution. We call it *two-sided generalized Weibull distribution* and denote it by $TSGW(\alpha,\gamma,\theta,\beta)$. The parameters α and γ are the shape parameters, β is the reflection parameter and θ is the scale parameter of the distribution. While the contribution of the first piece of the pdf is $1-\beta$, the contribution of the other part is β. Further, the *TSGW* distribution is in fact a mixture of the *EW* distribution truncated above at $\theta(-\log\beta)^{1/\gamma}$ and the $W(\theta\alpha^{-1/\gamma},\gamma)$ distribution truncated below at the same point, with the mixing parameter β, that is,

$$TSGW(\alpha,\gamma,\theta,\beta) = (1-\beta)EW_{(0,\eta)}(\alpha,\gamma,\theta) + \beta W_{(\eta,\infty)}(\theta\alpha^{-1/\gamma},\gamma), \tag{2.2}$$

where $W_{(a,b)}$ denotes the doubly truncated Weibull distribution with truncation points a and b, and similarly for $EW_{(a,b)}$.

2.2. *Density Shape*

The $TSGW$ distribution becomes very different forms by varying the two shape parameters. The density shape analysis of the distribution is given below. When $x > \eta$, $(d\log f)/dx = (\gamma - 1)/x - \alpha\gamma\theta^{-\gamma}x\gamma - 1$. Clearly, for $\gamma \leq 1$, the pdf is the decreasing function on this part. For $\gamma > 1$, the root of this derivative is $x^* = \theta((\gamma-1)/\alpha\gamma)^{1/\gamma}$ and at this point the value of the second derivative is $(d^2\log f)/dx^2 = -(\gamma-1)/(x^*)^2 - \alpha\gamma(\gamma-1)\theta^{-\gamma}(x^*)^{\gamma-2} < 0$. So, the function is unimodal (log-concave) and x^* is the mode of the pdf on this part. But, when $x^* < \eta$ for $\gamma > 1$, the pdf is again decreasing (See Figure 1). On the second part of the support of the distribution, that is when $x < \eta$, $(d\log f)/dx = (\gamma-1)/x - (1-\alpha)\gamma x^{\gamma-1}e^{-(x/\theta)^\gamma}\theta^{-\gamma}(1-e^{-(x/\theta)^\gamma})^{-1} - \gamma x^{\gamma-1}\theta^{-\gamma}$. Thus, the pdf is decreasing for $\gamma \leq 1$ and $\alpha \leq 1$. Otherwise, the mode is the solution of the following nonlinear equation

$$\alpha\gamma\left(\frac{x}{\theta}\right)^\gamma + \gamma e^{(x/\theta)^\gamma}\left[1 - \left(\frac{x}{\theta}\right)^\gamma\right] - \gamma - e^{(x/\theta)^\gamma} + 1 = 0.$$

Since the $TSGW$ distribution is the mixture of the truncated EW and the truncated W distributions, some shape properties of the distribution inherit from the known results in the literature. According to [15], the EW distribution is the decreasing one for $\alpha\gamma \leq 1$, and increasing one for otherwise. Hence, the $TSGW$ distribution is bimodal for $\alpha\gamma > 1$ provided that $\gamma > 1$. Also, it is unimodal for $\alpha\gamma \leq 1$ provided that $\gamma < 1$ (on the first part) and $\alpha\gamma \leq 1$ provided that $\gamma > 1$ (on the second part). The shapes of the pdf for selected parameter values are sketched in Figure 1.

Also, it can easily be seen that the right and left hand limits of the derivative of f at $x = \eta$ are equal to each other only when $\alpha = \gamma = 1$, which is the case of the exponential distribution. Otherwise, they are different and $f^{'}(\eta)$ does not exist. So the pdf has a corner point at η. On the other hand, since these limits will be equal for $\gamma \leq 1$ and when $\alpha \to 0$, this corner point will disappear. Similarly, the same will also be true when $\gamma \to 0$, for $\alpha\gamma \leq 1$ (See Figure 1 (a) and (b)).

Finally, we have the following limit cases from [22]. So the behavior of the pdf at the end points of the support is given by $\lim_{x\to\infty} f(x) = 0$ and

$$\lim_{x\to 0} f(x) = \begin{cases} 0, & \alpha\gamma > 1, \\ \theta^{-\gamma}, & \alpha\gamma = 1, \\ \infty, & \alpha\gamma < 1. \end{cases}$$

The scale parameter θ effects the tails of the distribution. So larger values of θ are associated with the thicker tails of the distribution.

2.3. *Special and Limiting Cases*

The $TSGW$ distribution contains many well-known distributions for special or limiting cases of the parameters. They are given in the following.

- For $\alpha = 1$, $TSGW$ is the ordinary Weibull distribution $W(\gamma, \theta)$,
- For $\alpha = \gamma = 1$, $TSGW$ becomes the ordinary exponential distribution with scale parameter θ,
- For $\alpha = 1$ and $\gamma = 2$, $TSGW$ becomes the Rayleigh distribution with scale parameter θ,
- For $\gamma = 1$, $TSGW$ is reduced to the two-sided generalized exponential distribution introduced by [7],

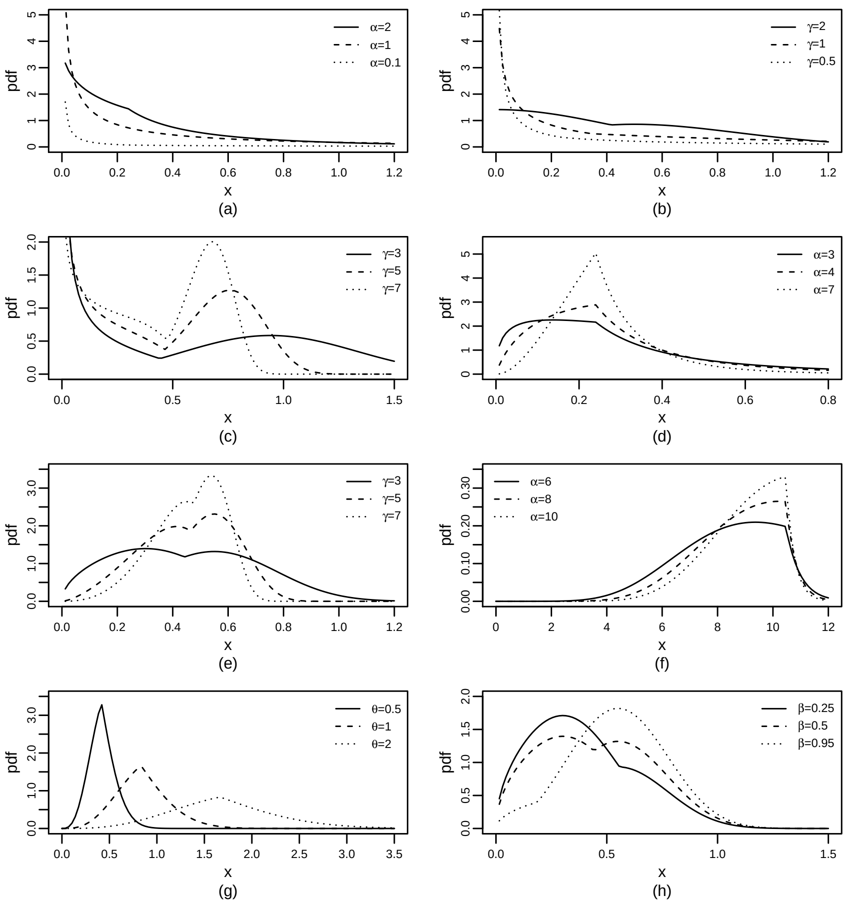

Fig. 1. The graphs of the pdf of the $TSGW$ with (a) $\gamma = \beta = \theta = 0.5$, (b) $\alpha = \beta = \theta = 0.5$, (c) $\alpha = 0.1$, $\beta = \theta = 0.5$, (d) $\gamma = \beta = \theta = 0.5$, (e) $\alpha = \beta = \theta = 0.5$, (f) $\gamma = 1.5$, $\theta = 6$, $\beta = 0.1$, (g) $\alpha = \gamma = 2$, $\beta = 0.5$, (h) $\gamma = 3$, $\alpha = \theta = 0.5$.

- For $\gamma = 2$, $TSGW$ is reduced to a distribution which we may call it the two-sided generalized Rayleigh distribution,
- When $\beta \to 0$, $EW(\alpha, \gamma, \theta)$ is obtained. It is also a special case of the beta modified Weibull distribution introduced by [29].
- When $\beta \to 0$ and $\gamma = 1$, the generalized exponential distribution of [2] is obtained.
- When $\beta \to 0$ and $\gamma = 2$, Burr type X distribution, also called the generalized Rayleigh distribution, is obtained.
- When $\beta \to 1$, we have $W(\theta\alpha^{-1/\gamma}, \gamma)$.

2.4. *Percentiles and Random Variate Generation*

The 100qth percentile x_q of the distribution is defined by $F(x_q; \alpha, \gamma, \beta, \theta) = q$ and is obtained as

$$x_q = \begin{cases} \theta\left\{-\log\left[1-\left(q(1-\beta)^{\alpha-1}\right)^{1/\alpha}\right]\right\}^{1/\gamma}, & 0 < q \le 1-\beta \\ \theta\left\{-\alpha^{-1}\log\left[(1-q)\beta^{\alpha-1}\right]\right\}^{1/\gamma}, & 1-\beta < q < 1. \end{cases}$$

A simple way of generating random variates from the distribution is performed by using the inverse transformation method. Accordingly, if U is a uniform random variate on (0,1), then

$$X = \begin{cases} \theta\left\{-\log\left[1-\left(U(1-\beta)^{\alpha-1}\right)^{1/\alpha}\right]\right\}^{1/\gamma}, & 0 < U \leq 1-\beta \\ \theta\left\{-\alpha^{-1}\log\left[(1-U)\beta^{\alpha-1}\right]\right\}^{1/\gamma}, & 1-\beta < U < 1 \end{cases}$$

has the $TSGW$ distribution through the probability integral transform. Also, the mixture form in Eq. (2.2) can be used to generate random variates from the distribution. This will be a two-stage process. We first select either truncated EW or truncated W, with proportions $(1-\beta)$ and β, respectively. Then a random number is generated from the selected distribution.

3. Moments

The moment generating function is the expectation $E[\exp(\theta t(-\log Y)^{1/\gamma})]$ which is obtained by a straightforward calculation, and is given by

$$M(t) = \beta^{1-\alpha}\sum_{k=0}^{\infty}\frac{(\theta t)^k\Gamma(k/\gamma+1,-\alpha\log\beta)}{\alpha^{k/\gamma}k!} + \frac{\alpha}{(1-\beta)^{\alpha-1}}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\binom{\alpha-1}{j}(\theta t)^k\gamma^*(k/\gamma+1,-(j+1)\log\beta)}{k!(j+1)^{k/\gamma+1}},$$

where $\Gamma(\cdot,\cdot)$ is the incomplete gamma function (see formula 3.381.3 in [1]) and

$$\gamma^*(\nu,u) = \int_0^u p^{\nu-1}e^{-p}\,dp,\ \nu > 0$$

is another incomplete gamma function (see formula 3.381.1 in [1]).

The rth moment of the $TSGW$ is given by

$$E(X^r) \equiv \mu_r = \frac{\theta^r\alpha}{(1-\beta)^{\alpha-1}}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha-1}{j}\frac{\gamma^*\left(\frac{r}{\gamma}+1,-(j+1)\log\beta\right)}{(j+1)^{r/\gamma+1}} + \frac{\theta^r}{\alpha^{r/\gamma}\beta^{\alpha-1}}\Gamma\left(\frac{r}{\gamma}+1,-\alpha\log\beta\right). \quad (3.1)$$

If r/γ is a positive integer, say n, then (3.1) is reduced to

$$\frac{\theta^r\alpha}{(1-\beta)^{\alpha-1}}\sum_{j=0}^{\infty}(-1)^j\binom{\alpha-1}{j}\frac{n!}{(j+1)^{n+1}}\left[1-\beta^{j+1}\sum_{m=0}^{n}\frac{(-1)^m[(j+1)\log\beta]^m}{m!}\right] + \frac{\theta^r\beta^\alpha n!}{\alpha^n\beta^{\alpha-1}}\sum_{m=0}^{n}\frac{(-\alpha\log\beta)^m}{m!}, \quad (3.2)$$

using the formulas 8.352.1 and 8.352.2 in [1].

For simplicity we may assume that the scale parameter $\theta = 1$ since if $X \sim TSGW(\alpha,\beta,\gamma)$ then $\theta X \sim TSGW(\alpha,\beta,\gamma,\theta)$. In Table (1), we calculated the mean, variance, median, coefficient of variation $CV = (\mu_2-\mu_1^2)^{1/2}/\mu_1$, measure of skewness $\delta_1 = (\mu_3-3\mu_1\mu_2+2\mu_1^3)/(\mu_2-\mu_1^2)^{3/2}$ and measure of kurtosis $\delta_2 = (\mu_4-4\mu_1\mu_3+6\mu_2\mu_1^2-3\mu_1^4)/(\mu_2-\mu_1^2)^2$ for selected parameters using (3.1). For $\beta = 0.5$ the median of the $TSGW$ is equal to that of Weibull distribution, regardless of the values

Table 1. The mean, variance, median (m), coefficient of variation (CV), skewness (δ_1) and kurtosis (δ_2) of $TSGW$ for some selected parameter values.

β	α	γ	$E(X)$	$V(X)$	m	CV	δ_1	δ_2
0.1	0.5	0.5	2.7073	114.3693	0.1058	3.9501	11.1385	237.8726
0.1	0.5	2	0.7109	0.3468	0.5704	0.8283	1.3524	5.1089
0.1	0.5	5	0.7906	0.5704	0.7988	0.3904	-0.1889	3.1111
0.1	2	0.5	2.1433	6.7598	1.2346	1.2130	2.9256	19.8121
0.1	2	2	1.0584	0.1237	1.0541	0.3323	0.0865	2.6206
0.1	2	5	1.0079	0.0206	1.0213	0.1426	-0.4799	3.0868
0.1	4	0.5	2.6602	3.7038	2.2518	0.7234	1.0812	4.8599
0.1	4	2	1.2052	0.0647	1.2250	0.2111	-0.3269	2.6685
0.1	4	5	1.0712	0.0091	1.0845	0.0891	-06857	3.3320
0.25	0.5	0.5	4.0544	175.5937	0.1644	3.2683	8.9553	155.8528
0.25	0.5	2	0.8129	0.4443	0.6367	0.8199	1.1163	3.9541
0.25	0.5	5	0.8332	0.1072	0.8348	0.3920	-0.1912	2.8550
0.25	2	0.5	1.4731	3.5233	0.8981	1.2742	3.9256	34.3274
0.25	2	2	0.9705	0.0958	0.9735	0.3190	0.1401	3.0049
0.25	2	5	0.9448	0.0176	0.9893	0.1362	-0.4894	3.3586
0.25	4	0.5	1.4351	0.9918	1.2820	0.6939	1.7795	10.6898
0.25	4	2	1.0436	0.0387	1.0640	0.1886	-0.3048	3.2535
0.25	4	5	1.0125	0.0064	1.0251	0.0791	-0.7207	3.9028
0.5	0.5	0.5	5.6655	239.4315	0.4804	2.7311	7.6472	115.2935
0.5	0.5	2	0.9652	0.5149	0.8325	0.7345	0.7687	3.0891
0.5	0.5	5	0.9016	0.1123	0.9293	0.3717	-0.3879	2.7903
0.5	2	0.5	0.9401	2.1061	0.4804	1.5436	5.1455	54.9894
0.5	2	2	0.8520	0.0808	0.8325	0.3337	0.4798	3.5543
0.5	2	5	0.9248	0.0164	0.9293	0.1387	-0.2321	3.4170
0.5	4	0.5	0.6192	0.3040	0.4804	0.8903	3.5141	28.2507
0.5	4	2	0.8390	0.0273	0.8325	0.1971	0.3402	4.0107
0.5	4	5	0.9277	0.0055	0.9293	0.0799	-0.1786	3.9664
0.75	0.5	0.5	6.9289	284.5438	1.2069	2.4344	7.0125	97.8431
0.75	0.5	2	1.1087	0.5090	1.0481	0.6434	0.5563	2.9601
0.75	0.5	5	0.9719	0.1004	1.0189	0.3261	-0.6491	3.2841
0.75	2	0.5	0.6625	1.5608	0.2405	1.8855	5.9519	71.8038
0.75	2	2	0.7440	0.0833	0.7002	0.3879	0.7517	3.6752
0.75	2	5	0.8724	0.0188	0.8671	0.1575	0.0642	3.0317
0.75	4	0.5	0.2769	0.1448	0.1513	1.3745	4.9878	51.4105
0.75	4	2	0.6555	0.0295	0.6237	0.2622	0.9161	4.2610
0.75	4	5	0.8379	0.0074	0.8279	0.1026	0.4141	3.3878

of α and γ. The mean and median values decrease for fixed large α and fixed γ for increasing β. On the other hand, for fixed α and fixed β (γ), skewness is negative for increasing γ (β). The distribution has large kurtosis for small values of α and γ, while it has small kurtosis for increasing α and γ. When $\gamma = 2$ and $\beta = 0.5$, the kurtosis of distribution is first decreasing and then increasing. When γ increases for its large values, the mean increases for fixed α and β, however the variance and CV decrease. For small values of α and fixed γ, the mean and the median increase for increasing β. For large α and γ, the variance gets smaller. Also, the variance increases for fixed and small α and γ

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for increasing β. In addition, for small β and fixed γ, we obtain negatively skewed distributions for increasing α (see Figure 2).

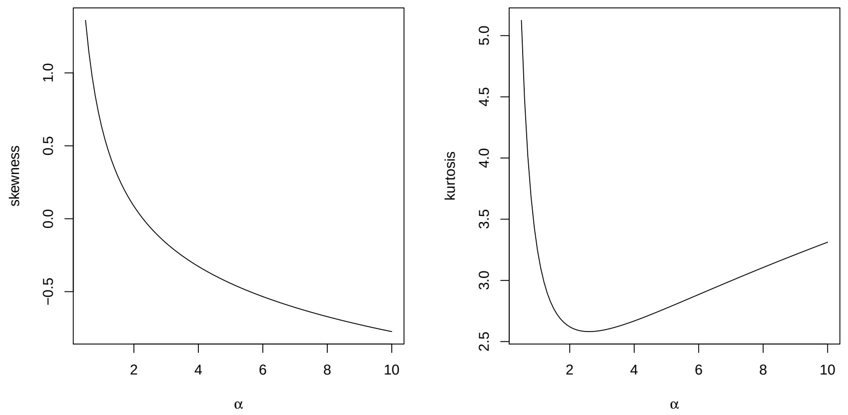

Fig. 2. Plots of the skewness and kurtosis values of the $TSGW$ for $\gamma = 2$ and $\beta = 0.1$

4. Hazard Rate Function

The hazard function $r(t)$ is very important in lifetime studies owing to the diversity of the given data. Since the distribution is two-sided, it can be more useful to model hazard shapes with different characteristics. The hazard function of the distribution is given by

$$r(t) = \begin{cases} \dfrac{\alpha\gamma(1-\beta)^{1-\alpha} t^{\gamma-1}\left[1-\exp\left(-(\frac{t}{\theta})^{\gamma}\right)\right]^{\alpha-1}\exp\left(-(\frac{t}{\theta})^{\gamma}\right)}{\theta^{\gamma}\left\{1-(1-\beta)^{1-\alpha}\left[1-\exp\left(-(\frac{t}{\theta})^{\gamma}\right)\right]^{\alpha}\right\}}, & t \leq \eta, \\ \dfrac{\alpha\gamma t^{\gamma-1}}{\theta^{\gamma}}, & t > \eta. \end{cases}$$

The shape of the hazard function becomes different on the parts of the support. We see that on (η, ∞) the hazard rate of distribution is the same with that of Weibull distribution. In that case, the hazard function can be constant, decreasing or increasing for $\gamma = 1$, $\gamma > 1$ and $\gamma > 1$, respectively. On the other part, the hazard function has the same shapes as that of the EW distribution which can be found in [15]. Therefore, we may state the overall shape properties of the hazard function of the $TSGW$ distribution in the following theorem.

Theorem 4.1. *$r(t)$ of the $TSGW$ distribution is*

- *monotone IHR (increasing hazard rate) function throughout its support if $\gamma > 1$ and $\alpha\gamma \geq 1$,*
- *monotone DHR (decreasing hazard rate) function throughout its support if $\gamma \leq 1$ and $\alpha\gamma < 1$,*
- *firstly IHR (DHR) then constant if $\gamma = 1$ and $\alpha\gamma \geq 1$ ($\alpha\gamma \leq 1$),*
- *firstly BHR (bathtube hazard rate) then IHR function if $\gamma > 1$ and $\alpha\gamma < 1$,*
- *UHR (unimodal hazard rate) then DHR function, if $\gamma < 1$ and $\alpha\gamma > 1$,*

- *constant hazard rate function when* $\alpha = \gamma = 1$.

We note that the shapes of the hazard function depend on α and γ on the first part, but it only depends on γ on the second part. Also, we have the following limiting cases of $r(t)$.

$$\lim_{t\to 0} r(t) = \begin{cases} 0, & \alpha > 1, \gamma > 1, \\ \theta^{-1}, & \alpha = 1, \gamma = 1, \\ \infty, & \alpha < 1, \gamma < 1. \end{cases}$$

$$\lim_{t\to \infty} r(t) = \begin{cases} 0, & \gamma < 1, \\ \theta^{-1}, & \alpha = 1, \gamma = 1, \\ \infty, & \gamma > 1. \end{cases}$$

Thus, while $\gamma > 1$ and $\alpha > 1$ the hazard function increases from 0 to infinity. On the other hand, for $\gamma < 1$ and $\alpha\gamma < 1$, the hazard function is non-increasing (See Figure 3).

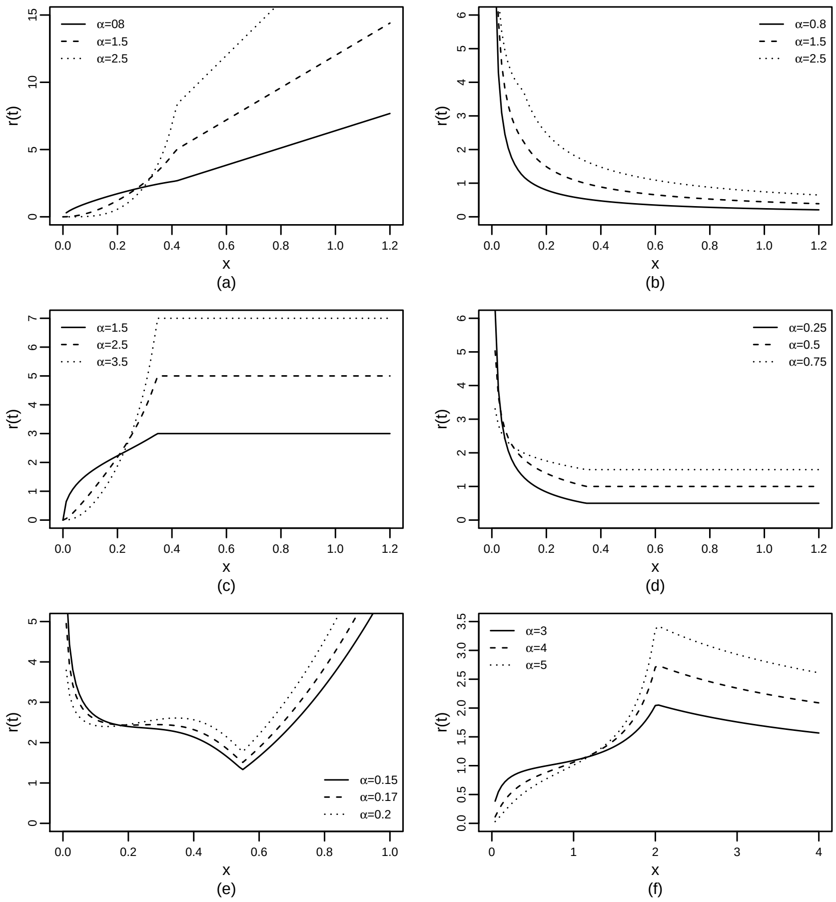

Fig. 3. The graphs of the hazard rate of the *TSGW* with (a) $\gamma = 2$, $\beta = \theta = 0.5$, (b) $\gamma = 0.25$, $\beta = \theta = 0.5$, (c) $\gamma = 1$, $\beta = \theta = 0.5$, (d) $\gamma = 1$, $\beta = \theta = 0.5$, (e) $\beta = 0.25$, $\theta = 0.5$, $\gamma = 3.5$ (f) $\gamma = 0.6$, $\theta = 0.5$, $\beta = 0.1$.

5. Rényi Entropy

The entropy of a random variable X is a measure of variation of the uncertainty. The Rényi entropy of the distribution with pdf $f(.)$ is given by the following integral

$$J_R(\sigma) = \frac{1}{1-\sigma} \log \int_{-\infty}^{\infty} f^{\sigma}(x)\, dx.$$

for $\sigma > 0$ and $\sigma \neq 1$. [18] derived the entropies for several univariate distributions including the TSP. Using Eq. (2.1), we obtain the Rényi entropy for the $TSGW$ distribution as

$$J_R(\sigma) = \frac{1}{1-\sigma} \left\{ \sigma \log \alpha + (\sigma-1)\log(\gamma/\theta) + \log \left[\frac{\Gamma[(\sigma-1)(\gamma-1)/\gamma+1, -\sigma\alpha \log\beta]}{\beta^{\sigma(\alpha-1)}(\sigma\alpha)^{(\sigma-1)(\gamma-1)/\gamma+1}} \right.\right.$$
$$\left.\left. + \frac{1}{(1-\beta)^{\sigma(\alpha-1)}} \sum_{j=0}^{\infty} (-1)^j \binom{\sigma(\alpha-1)}{j} \frac{\gamma^*((\sigma-1)(\gamma-1)/\gamma+1, -(\sigma+j)\log\beta)}{(\sigma+j)^{(\sigma-1)(\gamma-1)/\gamma+1}} \right] \right\}$$

provided that $(\sigma-1)(\gamma-1) \geq 0$.

6. Estimation

6.1. *Maximum Likelihood Estimation and an Algorithm*

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the $TSGW(\alpha,\gamma,\theta,\beta)$ and let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ denote the corresponding order statistics. Then the log-likelihood functions is given by

$$l(\alpha,\gamma,\theta,\beta) = n\log\alpha + n\log\gamma - \gamma n\log\theta + (\gamma-1)\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\gamma}$$
$$+ (\alpha-1)\log\left\{ \frac{\prod_{i=1}^{r}\left[1-\exp\left(-\left(\frac{x_{(i)}}{\theta}\right)^{\gamma}\right)\right]\prod_{i=r+1}^{n}\exp\left(-\left(\frac{x_{(i)}}{\theta}\right)^{\gamma}\right)}{(1-\beta)^r\beta^{n-r}} \right\}, \quad (6.1)$$

where $x_{(r)} \leq \eta < x_{(r+1)}$ for $r = 1,2,\ldots,n$ and $x_{(0)} \equiv 0$, $x_{(n+1)} \equiv \infty$.

The maximum likelihood estimates of the parameters maximize (6.1) globally. Note that we must also estimate r which is implicitly defined above. We will first consider the estimates of α and β. Taking the partial derivatives of (6.1) with respect to α and β, and then equating them to 0, we get

$$\hat{\alpha} = -\frac{n}{\log M(\hat{r},\gamma,\theta)},$$

$$\hat{\beta} = \exp\left[-\left(\frac{x_{(\hat{r})}}{\theta}\right)^{\gamma}\right],$$

where $\hat{r} = \arg\max_{r\in\{1,2,\ldots,n\}} M(r,\gamma,\theta)$ with

$$M(r,\gamma,\theta) = \prod_{i=1}^{r-1}\left[\frac{1-\exp\left(-\left(\frac{x_{(i)}}{\theta}\right)^{\gamma}\right)}{1-\exp\left(-\left(\frac{x_{(r)}}{\theta}\right)^{\gamma}\right)}\right]\prod_{i=r+1}^{n}\left[\frac{\exp\left(-\left(\frac{x_{(i)}}{\theta}\right)^{\gamma}\right)}{\exp\left(-\left(\frac{x_{(r)}}{\theta}\right)^{\gamma}\right)}\right].$$

We will need an iterative procedure to find the estimates. The associated likelihood estimating equations for the other parameters are given by

$$\frac{\partial l}{\partial \gamma} = \frac{n}{\gamma} - n\log\theta + \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\gamma}\log\left(\frac{x_i}{\theta}\right) + (\alpha-1)$$
$$\times \sum_{i=1}^{r}\frac{\left(\frac{x_{(i)}}{\theta}\right)^{\gamma}\exp\left[-(\frac{x_{(i)}}{\theta})^{\gamma}\right]\log\left(\frac{x_{(i)}}{\theta}\right)}{1-\exp\left[-\left(\frac{x_{(i)}}{\theta}\right)^{\gamma}\right]} - (\alpha-1)\sum_{i=r+1}^{n}\left(\frac{x_{(i)}}{\theta}\right)^{\gamma}\log\left(\frac{x_{(i)}}{\theta}\right) = 0, \qquad (6.2)$$

$$\frac{\partial l}{\partial \theta} = \frac{-\gamma n}{\theta} + \frac{\gamma}{\theta^{\gamma+1}}\sum_{i=1}^{n}x_i^{\gamma} - \frac{\gamma(\alpha-1)}{\theta^{\gamma+1}}\left\{\sum_{i=1}^{r}\frac{x_{(i)}^{\gamma}\exp\left[-(\frac{x_{(i)}}{\theta})^{\gamma}\right]}{1-\exp\left[-(\frac{x_{(i)}}{\theta})^{\gamma}\right]} - \sum_{i=r+1}^{n}x_{(i)}^{\gamma}\right\} = 0. \qquad (6.3)$$

To compute the estimates iteratively we may give the following algorithm.
Step 1: Set $k = 0$ and put initial estimates $\hat{\gamma}^{(0)}$ and $\hat{\theta}^{(0)}$ for γ and θ in the log likelihood l.
Step 2: Compute estimates

$$\hat{\alpha}^{(k+1)} = -\frac{n}{\log M(\hat{r},\hat{\gamma}^{(k)},\hat{\theta}^{(k)})},$$

$$\hat{\beta}^{(k+1)} = \exp\left[-\left(\frac{x_{(\hat{r})}}{\hat{\theta}^{(k)}}\right)^{\hat{\gamma}^{(k)}}\right],$$

where $\hat{r} = \underset{r\in\{1,2,\ldots,n\}}{\arg\max}\; M(r,\hat{\gamma}^{(k)},\hat{\theta}^{(k)})$ with

$$M(r,\hat{\gamma}^{(k)},\hat{\theta}^{(k)}) = \prod_{i=1}^{r-1}\left[\frac{1-\exp\left(-\left(\frac{x_{(i)}}{\hat{\theta}^{(k)}}\right)^{\hat{\gamma}^{(k)}}\right)}{1-\exp\left(-\left(\frac{x_{(r)}}{\hat{\theta}^{(k)}}\right)^{\hat{\gamma}^{(k)}}\right)}\right]\prod_{i=r+1}^{n}\left[\frac{\exp\left(-\left(\frac{x_{(i)}}{\hat{\theta}^{(k)}}\right)^{\hat{\gamma}^{(k)}}\right)}{\exp\left(-\left(\frac{x_{(r)}}{\hat{\theta}^{(k)}}\right)^{\hat{\gamma}^{(k)}}\right)}\right]. \qquad (6.4)$$

Step 3: Update γ and θ by using (6.2) and (6.3) to find $\hat{\gamma}^{(k+1)}$ and $\hat{\theta}^{(k+1)}$.
Step 4: If $|l(\hat{\alpha}^{(k+1)},\hat{\gamma}^{(k+1)},\hat{\theta}^{(k+1)},\hat{\beta}^{(k+1)}) - l(\hat{\alpha}^{(k+1)},\hat{\gamma}^{(k)},\hat{\theta}^{(k)},\hat{\beta}^{(k+1)})|$ is less than a tolerance level, say 10^{-2},
Stop
Else $k = k+1$ and Goto Step 2.

The expressions given in (6.2) and (6.3) can be easily differentiated, and hence the fixed point solutions of γ and θ in (6.2) and (6.3) can also be considered with the Newton method. With a good starting point of γ and θ, the convergence will hold. One should use a computer package such as MATLAB to write the codes of the algorithm.

It is well known that the maximum likelihood estimators are asymptotically unbiased and have an asymptotic normal distribution under some regularity conditions. The related information is contained in the Fisher information matrix which is the matrix whose elements are negative of expected values of the second partial derivatives of the log-likelihood function with respect to the parameters. Since these cannot be derived in a regular way due to the r which is implicitly defined, we investigate the bias properties empirically. So we perform a simulation study generating 10,000 samples

Table 2. Empirical means and standard deviations for different values of θ, β, γ and α.

				$n=20$				$n=100$			
θ	β	γ	α	$\hat{\alpha}(SD)$	$\hat{\gamma}(SD)$	$\hat{\beta}(SD)$	$\hat{\theta}(SD)$	$\hat{\alpha}(SD)$	$\hat{\gamma}(SD)$	$\hat{\beta}(SD)$	$\hat{\theta}(SD)$
1	0.4	0.5	1.5	1.7084(0.3874)	0.5296(0.0367)	0.3934(0.2545)	0.9937(0.2439)	1.5307(0.1484)	0.5119(0.0104)	0.3971(0.1139)	0.9963(0.1387)
1	0.4	0.5	2.5	2.7635(0.6419)	0.5436(0.0325)	0.4107(0.0888)	0.9530(0.2184)	2.5514(0.2781)	0.5083(0.0066)	0.4034(0.0386)	1.0132(0.0997)
1	0.4	1	1.5	1.6860(0.3858)	1.0562(0.0790)	0.4052(0.2672)	0.9723(0.1425)	1.5394(0.1657)	1.0321(0.0258)	0.4020(0.1303)	1.0081(0.0673)
1	0.4	1	2.5	2.8292(0.6563)	1.1007(0.0759)	0.3865(0.0825)	0.9937(0.1170)	2.5537(0.2455)	1.0213(0.0153)	0.4055(0.0455)	1.0011(0.0497)
1	0.4	2	1.5	1.7668(0.4027)	2.1007(0.1293)	0.3883(0.2506)	1.0022(0.0700)	1.5131(0.1482)	2.0617(0.0454)	0.4121(0.1282)	1.0022(0.0316)
1	0.4	2	2.5	2.7369(0.6292)	2.1645(0.1063)	0.3980(0.1014)	0.9930(0.0488)	2.5304(0.2552)	2.0198(0.0264)	0.4004(0.0320)	0.9990(0.0223)
1	0.5	0.5	1.5	1.7444(0.4244)	0.5280(0.0381)	0.4550(0.2603)	0.8512(0.4419)	1.5193(0.1465)	0.5144(0.0145)	0.4991(0.1388)	0.9498(0.1772)
1	0.5	0.5	2.5	2.7211(0.6231)	0.5297(0.0358)	0.5180(0.0956)	0.8623(0.2623)	2.5683(0.2854)	0.5076(0.0125)	0.5047(0.0406)	0.9970(0.1229)
1	0.5	1	1.5	1.7031(0.3973)	1.0403(0.0686)	0.4966(0.2601)	0.9323(0.1577)	1.5763(0.1654)	1.0221(0.0281)	0.4942(0.1056)	0.9816(0.0636)
1	0.5	1	2.5	2.6325(0.5086)	1.0889(0.0573)	0.5051(0.0966)	0.9428(0.1113)	2.5234(0.2792)	1.0037(0.0222)	0.4997(0.0375)	0.9854(0.0470)
1	0.5	2	1.5	1.7603(0.4175)	2.1246(0.1382)	0.4824(0.2492)	0.9897(0.0602)	1.5356(0.1488)	2.0505(0.0634)	0.5265(0.1309)	0.9799(0.0334)
1	0.5	2	2.5	2.6730(0.5881)	2.0852(0.1113)	0.5112(0.0916)	0.9694(0.0497)	2.5546(0.2374)	2.0092(0.0435)	0.5021(0.0357)	0.9833(0.0251)
1	0.6	0.5	1.5	1.6947(0.3423)	0.5122(0.0384)	0.5993(0.2512)	0.9112(3.2997)	1.5314(0.1524)	0.5081(0.0166)	0.5992(0.1289)	0.9688(0.2116)
1	0.6	0.5	2.5	2.8127(0.5967)	0.5018(0.0404)	0.6045(0.0869)	0.9285(1.7427)	2.5766(0.2827)	0.5105(0.0162)	0.5976(0.0366)	0.9772(0.1043)
1	0.6	1	1.5	1.7269(0.4195)	1.0124(0.0816)	0.6278(0.2390)	0.9136(0.1440)	1.5288(0.1521)	1.0090(0.0388)	0.6008(0.1314)	0.9649(0.0874)
1	0.6	1	2.5	2.6862(0.6442)	1.0344(0.0669)	0.6057(0.1079)	0.9000(0.1592)	2.5794(0.2872)	0.9919(0.0345)	0.6001(0.0369)	0.9698(0.0529)
1	0.6	2	1.5	1.7254(0.4448)	2.0409(0.1412)	0.6060(0.2418)	0.9424(0.0746)	1.5291(0.1458)	2.0161(0.0607)	0.6007(0.1094)	0.9847(0.0325)
1	0.6	2	2.5	2.7763(0.6867)	2.0202(0.1612)	0.5929(0.0901)	0.9362(0.0618)	2.5160(0.2401)	2.0028(0.0593)	0.5985(0.0379)	0.9879(0.0257)
2	0.4	0.5	1.5	1.7113(0.4160)	0.5289(0.0376)	0.3970(0.2703)	1.9245(0.6078)	1.5678(0.1657)	0.5153(0.0112)	0.3976(0.1172)	2.0310(0.2938)
2	0.4	0.5	2.5	2.7724(0.7403)	0.5598(0.0439)	0.4110(0.1041)	2.0316(0.4432)	2.5917(0.2586)	0.5053(0.0071)	0.4026(0.0337)	2.0084(0.1898)
2	0.4	1	1.5	1.6987(0.3785)	1.0560(0.0680)	0.4262(0.2806)	1.9500(0.3270)	1.5199(0.1550)	1.0393(0.0249)	0.4253(0.1294)	2.0230(0.1204)
2	0.4	1	2.5	2.6653(0.5777)	1.0782(0.0571)	0.4184(0.0915)	1.9756(0.2064)	2.5343(0.2752)	1.0222(0.0173)	0.4056(0.0377)	2.0239(0.0981)
2	0.4	2	1.5	1.7628(0.3726)	2.1233(0.1312)	0.4011(0.2310)	2.0021(0.1234)	1.5498(0.1538)	2.0632(0.0431)	0.4125(0.1273)	2.0101(0.0633)
2	0.4	2	2.5	2.7710(0.5894)	2.2356(0.1685)	0.3941(0.0980)	1.9946(0.0958)	2.5752(0.2475)	2.0504(0.0415)	0.4106(0.0372)	2.0012(0.0476)
2	0.5	0.5	1.5	1.7248(0.3388)	0.5154(0.0410)	0.4950(0.2819)	1.9080(4.6528)	1.5320(0.1428)	0.5081(0.0132)	0.4982(0.1109)	1.9230(0.2751)
2	0.5	0.5	2.5	2.7481(0.5455)	0.5392(0.0357)	0.5113(0.0976)	1.8075(0.5984)	2.5294(0.2796)	0.5101(0.0133)	0.4999(0.0399)	1.9924(0.2053)
2	0.5	1	1.5	1.6774(0.3753)	1.0447(0.0697)	0.5098(0.2806)	1.9479(0.2928)	1.5523(0.1507)	1.0380(0.0268)	0.4873(0.1181)	1.9764(0.1194)
2	0.5	1	2.5	2.8816(0.7418)	1.0582(0.0608)	0.5046(0.0847)	1.9488(0.2098)	2.5548(0.2554)	1.0181(0.0269)	0.5019(0.0431)	1.9729(0.1194)
2	0.5	2	1.5	1.7731(0.3881)	2.1186(0.1630)	0.4888(0.2502)	1.9698(0.1336)	1.5258(0.1457)	2.0492(0.0503)	0.4992(0.1153)	1.9857(0.0642)
2	0.5	2	2.5	2.7038(0.6155)	2.1804(0.1371)	0.4953(0.0962)	1.9837(0.1079)	2.5178(0.2328)	2.0245(0.0491)	0.4981(0.0349)	2.0058(0.0473)
2	0.6	0.5	1.5	1.7234(0.4023)	0.5124(0.0405)	0.5672(0.2653)	1.8246(4.7764)	1.5543(0.1637)	0.5040(0.0185)	0.5995(0.1164)	1.9659(0.2412)
2	0.6	0.5	2.5	2.8328(0.7888)	0.5187(0.0321)	0.5908(0.0959)	1.8871(4.0879)	2.5426(0.2487)	0.5018(0.0156)	0.5997(0.0366)	1.9696(0.2206)
2	0.6	1	1.5	1.6860(0.3303)	1.0144(0.0861)	0.6025(0.2681)	1.8396(0.3314)	1.5277(0.1291)	1.0034(0.0349)	0.5993(0.1121)	1.9339(0.1483)
2	0.6	1	2.5	2.7641(0.6544)	1.0090(0.0637)	0.5854(0.0921)	1.8193(0.2430)	2.5198(0.2603)	0.9978(0.0362)	0.6002(0.0376)	1.9459(0.1131)
2	0.6	2	1.5	1.7705(0.4324)	2.0254(0.1880)	0.6277(0.2322)	1.8887(0.1532)	1.5272(0.1607)	2.0014(0.0678)	0.6023(0.1291)	1.9728(0.0721)
2	0.6	2	2.5	2.7843(0.5960)	2.0288(0.1461)	0.5828(0.0816)	1.8996(0.1270)	2.5224(0.2564)	2.0032(0.0684)	0.5986(0.0375)	1.9882(0.0563)

of sizes 20 and 100 from the distribution. The results of the simulation are reported in Table (2). We observe that the estimates approaches to true values as the sample sizes increase.

7. Data Analysis

In this section, we demonstrate to use of the $TSGW$ and compare it with some generalized Weibull distributions on two real data sets. The first data set is from [23] and it consists of 100 observations on breaking stress of carbon fibres (in GPa). The data are: 0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.80, 1.84, 1.84, 1.87, 1.89, 1.92, 2.00, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.60, 3.65, 3.68, 3.68, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90, 4.91, 5.08, 5.56. [12] introduced the exponentiated generalized inverse Gaussian distribution, fitted it to this data set and compared the result to the fits of several models. [28] derived Fisher information matrix for the EW distribution under type II censoring and used this data set as an application.

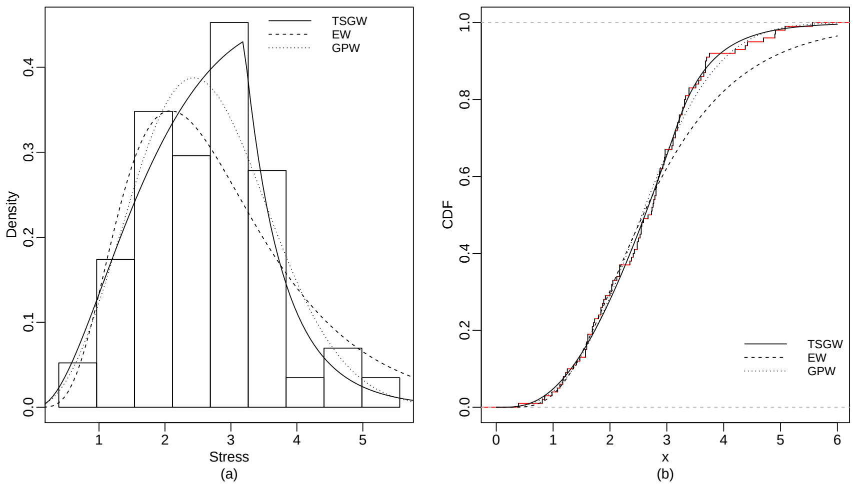

Fig. 4. (a) Histogram of the carbon data set and the superimposed fits (b) Empirical and fitted cdf's.

The second data are from the New York State Department of Conservation correspond to the daily ozone level measurements in New York in May-September, 1973. The data are: 41, 36, 12, 18, 28, 23, 19, 8, 7, 16, 11, 14, 18, 14, 34, 6, 30, 11, 1, 11, 4, 32, 23, 45, 115, 37, 29, 71, 39, 23, 21, 37, 20, 12, 13, 135, 49, 32, 64, 40, 77, 97, 97, 85, 10, 27, 7, 48, 35, 61, 79, 63, 16, 80, 108, 20, 52, 82, 50, 64, 59, 39, 9, 16, 78, 35, 66, 122, 89, 110, 44, 28, 65, 22, 59, 23, 31, 44, 21, 9, 45, 168, 73, 76, 118, 84, 85, 96, 78, 73, 91, 47, 32, 20, 23, 21, 24, 44, 21, 28, 9, 13, 46, 18, 13, 24, 16, 13, 23, 36, 7, 14, 30, 14, 18, 20. This data set is apparently more skewed than the carbon data set. Recently, [20] and [11] analyzed these data using a truncated version of inverted beta and an extended Birnbaum-Saunders distributions, respectively.

To see the performance of the $TSGW$, we fit it to both of these data sets. We also fit two Weibull extensions: the exponentiated Weibull (EW) whose pdf is given in Introduction Section and the

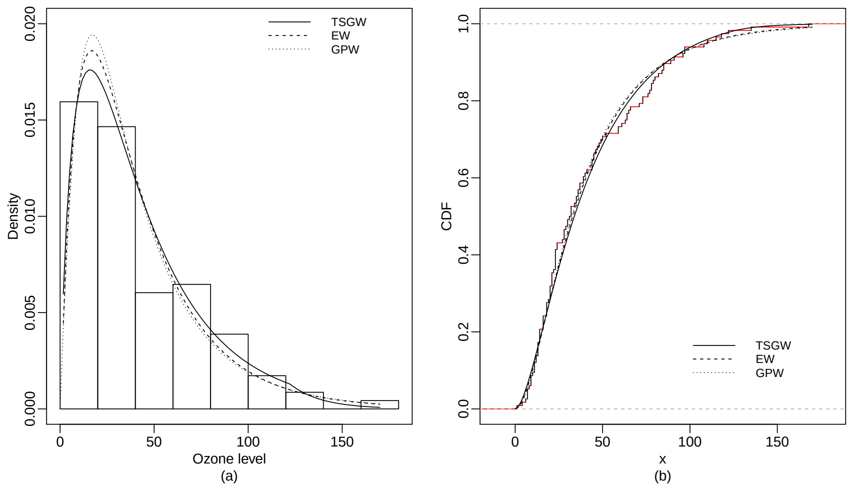

Fig. 5. (a) Histogram of the ozone data set and the superimposed fits (b) Empirical and fitted cdf's.

Table 3. Maximum likelihood parameter estimates, log-likelihood, AIC and $K-S$ values of the $TSGW$ and some other existing methods for the Carbon data set. (The standard errors for the estimates and p-values of the $K-S$ are given in parentheses).

Model	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\beta}$	l	AIC	K-S
EW	6.8091	0.9628	1.0702		-148.0784	302.157	0.1370
	(2.1364)	(0.0193)	(0.0224)				(0.0426)
GPW	1.3212	3.0689	2.5561		-141.3300	288.660	0.0644
	(0.5369)	(0.5110)	(0.5149)				(0.8006)
$TSGW$	6.3043	0.6224	0.2615	1.9935	-139.9130	287.826	0.0444
	(0.1823)	(0.0014)	(0.0071)	(0.0051)			(0.9875)

generalized power Weibull (GPW) ([24]) with pdf

$$f_{GPW}(x;\alpha,\gamma,\theta) = \frac{\gamma}{\alpha\theta^{\gamma}}x^{\gamma-1}\left[1+\left(\frac{x}{\theta}\right)^{\gamma}\right]^{1/\alpha-1}\exp\left\{1-\left[1+\left(\frac{x}{\theta}\right)^{\gamma}\right]^{1/\alpha}\right\},$$

where $x,\alpha,\gamma,\theta > 0$. They have both two shape parameters and one scale parameter and thus they are two natural competitors for the $TSGW$.

We apply the MLE procedure and use the algorithm given above for computations of the estimates obtained from the $TSGW$ model. Tables (3) and (4) report the MLEs (and the corresponding standard errors) of the model parameters of our model and its competitors with corresponding log-likelihood values, AIC (Akaike Information Criterion) and $K-S$ (Kolmogorov-Smirnov) test statistic values for the data sets. We observe from Tables (3) and (4) that the $TSGW$ distribution has the smallest AIC values. So it could be chosen as the best model among the other models under this criteria. The Figures 4 and 5 of the fitted densities and their empirical cdf's also support this observation. The $TSGW$ fit successfully and nicely captures the peak. Also, our proposed model performs better for the second data set than the other two distributions.

Table 4. Maximum likelihood parameter estimates, log-likelihood, AIC and $K-S$ values of the $TSGW$ and some other existing methods for the Ozone data set. (The standard errors for the estimates and p-values of the $K-S$ are given in parentheses).

Model	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\beta}$	l	AIC	K-S
EW	2.5805	0.8350	21.1821		-541.203	1088.405	0.0750
	(1.6124)	(0.2429)	(13.1193)				(0.5310)
GPW	2.3997	1.9158	19.849		-541.118	1088.236	0.0698
	(0.9753)	(0.4256)	(6.5940)				(0.6246)
$TSGW$	2.2314	0.8471	0.0219	25.1020	-536.5	1081.0	0.0797
	(0.0785)	(0.0182)	(0.0306)	(0.2862)			(0.4519)

8. Conclusions

We introduce a new generalization of Weibull distribution. Our methodology is based on a transformation of the standard two-sided power distributed random variate. We study its properties and use it to model some real data sets. The proposed model contains not only the ordinary Weibull distribution but also other some well-known generalized distributions, and it is proven that it is also useful for modeling lifetime data.

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