Markov Chain Approximation method for Pricing Barrier Options with Stochastic Volatility and Jump

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Abstract— The purpose of this paper is to provide an efficient pricing method for barrier option with stochastic volatility and jump risk. First, by constructing a nonuniform variance grid and using local consistency arguments, this paper approximates the stochastic volatility jump-diffusion model with a finite and dense Markov chain; Then, the paper computes the rate matrix of the Markov chain by solving a system induced by local consistency conditions; And then the paper provides the character function of the Markov chain. At last, using Markov chain approximation method and Fourier transform technique, the paper obtains numerical solutions for barrier options pricing. Numerical results show that comparing with the Monte Carlo simulation, the proposed pricing technique is accurate, fast and easy to implement.

Keywords—Barrier option; option pricing; Markon chain; stochastic volatility; jump diffusion

I. INTRODUCTION

Barrier options [1] are among the most popular path-dependent derivatives that disappear or appear if the underlying asset price crosses a given level (barrier level) before expiration date. Such contracts form effective risk management tools, and are liquidly traded in the foreign exchange markets. The most frequently used standard barrier options are knock in and knock out options. Knock in options can be divided into two categories, up and in, and down and in. Similarly, knocking out options can also be used as a down and out and up and out option. This paper focuses on standard knock out call options.

There exists currently a good deal of literature on numerical methods for the pricing of barrier options. It is well known that in this case a straightforward Monte Carlo simulation algorithm will be time-consuming and yield unstable results for the prices and especially the sensitivities. Since option pricing can be described as the solution of a partial differential equation (PDE) or partial integro-differential equation (PIDE) with boundary condition [2], some researchers have priced barrier options through PDE (PIDE) method under stochastic volatility model or jump-diffusion model [3-7]. However some empirical evidences [8-9] show that the model that combines stochastic volatility and jumps may be more reasonable. But a complex model with too many stochastic factors will lead to difficulty of obtaining the solution of the corresponding pricing equation.

A different approach, pioneered by Kushner [10], is the Markov chain approximation method. Originally developed for the numerical solution of stochastic optimal control problems in continuous time, this method consists of approximating the system of interest by a discrete time chain that closely follows its dynamics, and solving the problem of interest for this chain. An application to the pricing of American options under jump diffusion model is given in [11]. Zhang et al [12] consider lookback options pricing in a stochastic volatility model. Mijatović [13] prices barrier options in a local volatility jump diffusion model. However, there is rare study for valuation of barrier option under a stochastic volatility jump diffusion model, which is rather challenge due to the nonlinearity and jump discontinuity.

The rest of the paper is organized as follows. Section 2 develops the underlying pricing model. Section 3 describes Markon chain approximation method. Section 4 presents numerical results for barrier options pricing. Section 5 concludes the paper.

II. THE MODEL

An arbitrage-free, frictionless financial market is considered where only riskless asset B and risky asset S are traded continuously up to a fixed horizon date T . Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a complete probability space with a filtration satisfying the usual conditions, i.e. the filtration is continuous on the right. Suppose $W^1(t)$ and $W^2(t)$ are both standard Brownian motion which is $\mathcal{F}_t$ -adapted, and $W^1(t)$ has correlation ρ with $W^2(t)$ .Let $N(t)$ be independent Poisson process with constant intensity λ , $V(t)$ and $S(t)$ denote the volatility and price process of stock. According to [14], the stochastic volatility jump-diffusion model can be represented by the following:

$$\begin{cases} d\log S(t) = (r - \lambda\bar{\mu})dt + \sqrt{V(t)}dW^1(t) + \xi_s dN(t) \\ dV(t) = k(\theta - V(t))dt + \sigma\sqrt{V(t)}dW^2(t) \\ dW^1(t)W^2(t) = \rho dt \end{cases} \tag{1}$$

where r is constant interest rate, ξ_s is the jump size, and $\ln(\xi_s + 1)$ has an normal distribution with mean μ_J and variance $\sigma_J{}^2$. $\lambda\bar{\mu}$ denotes the drift term compensates for the expected drift added by the jump component. By the Itô formula, $\bar{\mu} = E(\xi_S) = \exp(\mu_J + \frac{1}{2}\sigma_J{}^2) - 1$. $V(t)$ is a square root mean reverting process, first proposed by Heston [15]. k, θ, σ The parameter $k > 0$ is the mean-reversion rate, $\theta > 0$ is the longterm mean, $\sigma > 0$ is the volatility-of-variance. In this paper, the author always assumes that $2k\theta > \sigma^2$, which is known as the Feller condition.

III. OPTION PRICING BASE D ON MARKON CHAIN APPROXIMATION METHOD

According to [16], the volatility process $V(t)$ can be approximated arbitrarily well using a carefully selected Markov chain which satisfies the local consistency requirements.

A. The "local consistency" concept

Suppose $V^h = \{V_0{}^h, V_1{}^h, \cdots, V_{N_h}{}^h\}$ is a variance grid where $h > 0$ denotes the spacing between discrete points. Assume that $|V^h(t+\delta) - V^h(t)| \to 0$, $N_h \to \infty$ and the grid can cover the domain of $V(t)$ as $h \to 0$. Suppose $V^h(t) \in V$ denote the approximating chain of the process $V(t)$, and the corresponding rate matrix is $Q^h = [q_{ij}^h]$.

Assume that the values of the variance process and the approximating chain coincid, $V^h(t) = V(t)$. Suppose E_t^h denotes the expectation. For some $\delta > 0$, $V^h(t)$ meets the following local consistency conditions [10]:

$$E_t^h\{V^h(t+\delta) - V^h(t)\} = k(\theta - V(t))\delta + o(\delta) \tag{2}$$

$$E_t^h\{V^h(t+\delta) - V^h(t)\}^2 = \sigma^2 V(t)\delta + o(\delta) \tag{3}$$

$$|V^h(t+\delta) - V^h(t)| = o(h) \tag{4}$$

B. Construction of the approximating Markov chain of the model

1) The grid

A suitable choice of the grid is essential for the effectiveness of the pricing algorithm. One of the features of a good grid is that it has sufficient resolution in regions of interest, such as the current spot value and the barrier levels, which is a necessary condition for constructing a Markov chain market model that approximates well the dynamics of the given price process. Another desirable feature is that the grid "covers" a sufficiently large part of the state-space, which is needed to control the truncation error that arises when approximating an infinite state-space by a finite state-space. To employ a uniform grid that satisfies these conditions would be computationally expensive. Here the author employs the following procedure for generating a suitable nonuniform grid G, based on an algorithm from [13]:

a) Pick $N_i \in N$ and $d_i^{\pm} \in (0, +\infty)$, $i = 1, 2, 3$ and the smallest and largest values V_1, V_N of the grid V , such that $N = N_1 + N_2 + N_3$.

b) Define the subgrid

$$G_i = G(a_i, s_i, b_i, N_i, d_i^-, d_i^+),\ i = 1, 2, 3.$$

where $a_1 = V_1$, $s_1 = l$, $s_2 = S_0$, $s_3 = u, b_1 = a_2, b_2 = a_3, b_3 = V_N$.

The subgrid G_i is generated by the following procedure:

- Compute $c_1 = \arcsin h(\frac{a-s}{g_1}),\ c_2 = \arcsin h(\frac{b-s}{g_2}).$
- Define the lower part of the grid by the formula
$$x_k = s + g_1 \sinh(c_1(1-(k-1)/(M/2-1))),$$
where $k \in \{1, \ldots, M/2\}$.
- Define the upper part of the grid by the formula
$$x_{k+M/2} = s + g_2 \sinh(c_2 2k/M),$$
where $k \in \{1, \ldots, M/2\}$.

c) $G = G_1 + G_2 + G_3$.

2) Computation of the rate matrix

Assume that at time t, the variance is equal to V_j^h . Over a time interval δ , there are three possibilities: remain at V_j^h , move up by d_U to $V_{j+1}^h = V_j + d_U$, or move down by d_D to $V_{j-1}^h = V_j - d_D$. Local consistency condition (2) and (3) can therefore be restated as

$$-q_{j,j-1}^h d_D\delta + q_{j,j+1}^h d_U\delta = (r - \lambda\bar{\mu})\delta + o(\delta), \tag{5}$$

$$q_{j,j-1}^h d_D{}^2\delta + q_{j,j+1}^h d_U{}^2\delta = V^h(t)\delta + o(\delta). \tag{6}$$

Solving the above system, it can be obtained

$$q_{j,j-1}^h = \frac{\sigma^2}{2h^2}\exp(-V_j{}^h) - \frac{k}{h}\left(\left(\theta - \frac{\sigma^2}{2k}\right)\exp(-V_j{}^h) - 1\right),$$

$$q_{j,j}^h = -\frac{\sigma^2}{h^2}\exp(-V_j{}^h),$$

$$q_{j,j+1}^h = \frac{\sigma^2}{2h^2}\exp(-V_j{}^h) + \frac{k}{h}\left(\left(\theta - \frac{\sigma^2}{2k}\right)\exp(-V_j{}^h) - 1\right).$$

3) The characteristic function

Using the decompositions $W^1 = \rho W^2 + \sqrt{1-\rho^2}Z$, where Z is a standard Brownian motion, independent of all other processes, Model (1) can be written as

$$d\log S(t) = (r - \lambda\bar{\mu})dt + \sqrt{V(t)}\rho dW^2(t) + \sqrt{V(t)}\sqrt{1-\rho^2}dZ + \xi_s dN(t) \quad (7)$$

$$dV(t) = k(\theta - V(t))dt + \sigma\sqrt{V(t)}dW^2(t) \quad (8)$$

Equation (8) implies that $dW^2 = \dfrac{dV(t) - k(\theta - V(t))dt}{\sigma\sqrt{V(t)}}$, which can be substituted into the first expression, and substitute $V(t)$ by $V^h(t)$, it can be obtained

$$d\log S(t) \approx (r - \lambda\bar{\mu})dt + \frac{\rho}{\sigma}(dV^h(t) - k(\theta - V^h(t))dt) + \sqrt{V(t)}\sqrt{1-\rho^2}dZ + \xi_s dN(t).$$

As [16] shows, the approximating characteristic function of $\log\dfrac{S(t)}{S(0)}$ is given by

$$\phi_i(u) = l^T \exp\{t\mathrm{B(u)}\}e_i$$

where l is an (n×1) vector of ones, and e_i is the i-th (n×1) unit vector. The matrix function B(u) has elements of the form:

$$\beta_{j,i} = \begin{cases} q^h_{j,j} + \Psi^h_j(u), & i = j \\ q^h_{j,i} + \Psi^h_{j\pm}(u), & i = j \pm 1 \\ 0, & otherwise. \end{cases}$$

where $\Psi^h_j(u) = i(r - \lambda\bar{\mu})u - \frac{1}{2}(1-\rho^2)V^h(t)u^2 + \lambda(\phi_J - 1)$,

$$\phi_J = \exp\left\{-\lambda\bar{\mu}iut + \lambda t\left[(1+\bar{\mu})^{iu}\exp\left(\frac{1}{2}\sigma_J^2 iu(iu-1)\right) - 1\right]\right\},$$

$$\Psi^h_{j\pm}(u) = \exp\left\{\pm i\frac{\rho}{\sigma}hu\right\}.$$

C. Barrier option pricing

Let $\log S = x$ and denote with $U(x,t)$ the value of the barrier option at time t. $U(x,t)$ can be computed by

$$U(x,t) = E[C(S(\tau_B),\tau_B)]$$

where $C(S(\tau_B),\tau_B)$ is a discounted payoff function and τ_B is the first hitting time of the given barrier level B by the underlying asset process $S(t)$. For down-and-out call barrier options the payoff $C(S(\tau_B),\tau_B)$ is defined by

$$C(S(\tau_B),\tau_B) = \begin{cases} e^{-rT}\max(S(T)-K,0), & \tau_B = T, \\ 0, & \tau_B < T. \end{cases}$$

where K is given exercise price at expiration date T.

Using Markon chain approximation method, $U(x,t)$ can be obtain by the recursive relationship

$$U(x,t) \approx \exp(-r\Delta t)E(U(X_{t+\Delta t}, t+\Delta t, s_{t+\Delta t}) \mid X_t = x, s_t = i)$$
$$= \exp(-r\Delta t)\sum_{j=1}^{N}\int_R U(y, t+\Delta t, j)$$
$$\times \mathrm{P}(X_{t+\Delta t} - x \in dy \mid s_t = i, s_{t+\Delta t} = j)\mathrm{P}(s_{t+\Delta t} = j \mid s_t = i)$$

If denote with $f(y)$ the log-return density over a time interval of Δt, then the above relationship can be written as

$$U(x,t,i) = \exp(-r\Delta t)\sum_{j=1}^{N}\int_R U(y, t+\Delta t, j) f(y-x, i, j).$$

$f(y,i,j)$ can be retrieved by taking the inverse Fourier transform of the characteristic function $\phi_i(u)$.

IV. NUMERICAL EXAMPLE

This section uses the method from Section 3 to price barrier options. This paper first evaluates barrier options using Markov chain approximation method. Then a comparison of the speed and accuracy between the Markov chain approximation and the exact Monte Carlo simulation proposed by [17] is provided. For comparison the default parameters are used in [17] and listed in Table Ⅰ.

TABLE I. DEFAULT PARAMETERS FOR BARRIER OPTIONS PRICING

Parameter	Value
Initial asset price	$S(0) = 100$
Intensity of the Poisson process	$\lambda = 0.11$
Volatility of volatility	$\sigma = 0.27$
Interest rate	$r = 0.0319$
Long-run variance	$\theta = 0.014$
Initial variance	$V(0) = 0.008836$
Mean reversion	$k = 3.99$
expectation of the jump size	$\bar{\mu} = -0.12$
Correlation between returns and volatility	$\rho = -0.79$
Maturity date	$T = 5$

Table Ⅱ lists the comparison of the two methods. Except for barrier level $B = 90$, other parameters are the same as the ones in Table 1. For Monte Carlo, number of simulation is 100000. For Markov chain approximation, number of grid is 200.

TABLE II. COMPARISON OF DOWN AND OUT CALL OPTIONS PRICES BETWEEN MARKOV CHAIN APPROXIMATION AND MONTE CARLO SIMULATION

Exercise price	Exact Monte Carlo simulation	Markov chain approximation
80	32.3423(0.0284)	32.3417
85	28.9055(0.0271)	28.9060
90	25.4688(0.0259)	25.4679
95	22.1107(0.0248)	22.1099
100	18.9709(0.0235)	18.9701
105	16.0827(0.0220)	16.0833
110	13.4641(0.0204)	13.4633
115	11.1232(0.0188)	11.1225
120	9.0646(0.0171)	9.0651

Note: Numbers in parentheses are standard errors for the estimates of options prices.

The numerical experiment shows that Markov chain approximation is considerably faster than the Monte Carlo simulation. For the pricing of down and out call options Markov chain approximation takes about 0.03 seconds, while Monte Carlo simulation takes about 9.1 seconds. Moreover, Table Ⅱ suggests the accuracy of the Markov chain approximation method. If the Monte Carlo is considered to be the benchmark, the relative percentage pricing differences of Markov chain approximation are all less than 0.09%.

V. CONCLUSION

This paper combines stock price jumps and stochastic volatility and considers a general jump-diffusion model for pricing barrier options. By Markov chain approximation method and Fourier transform technique, the paper obtains numerical solutions for barrier options pricing. Numerical results show that the proposed pricing technique is accurate, fast and easy to implement. The paper presents an efficient pricing method for barrier option with stochastic volatility and jump risk.

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