

# Left ( Right ) Continuous Maps in Orderly Topological Spaces

Fan, Linyuan\*

School of Statistics

Capital University of Economics and Business

Beijing, P.R.China

e-mail: [fanlinyuan@cueb.edu.cn](mailto:fanlinyuan@cueb.edu.cn)

\*Corresponding Author

**Abstract**—In this paper, we define ‘global’ and ‘local’ right or left continuous map on order topological spaces, also we elaborate connections and differences between them and the continuous maps in general cases.

**Keywords**—Orderly Topological space; global left(right) continuous maps; local left(right) continuous maps.

## I. INTRODUCTION

In mathematics analysis we define not only the continuous map but also the left and right continuity of the map. However, there is nothing specific about the continuity of map in general topological space.

In this paper, we consider how to define left and right continuous map, discuss the rationality in a general topological space. Noticing that there is no such definition about left (right) in general topological space, we introduce order firstly so as to define an orderly topological space. Then we define open set and neighborhood, which leads to left and right continuity. By existence of local left and right continuous map, we finally discuss the rationality of global continuous map in general topological spaces.

## II. TOPOLOGICAL SPACE AND ORDER

### A. Topological space

**Definition 2.1** Let  $X$  be a set, and  $\Gamma$  is a subset of  $X$  satisfying,

- $X, \Phi \in \Gamma$  if  $A, B \in \Gamma, A \cap B \in \Gamma$
- if  $\Gamma_1 \subset \Gamma$ , then  $\bigcup_{A \in \Gamma_1} A \in \Gamma$ ,

then  $\Gamma$  is a topology of  $X$ .

Assuming that  $\Gamma$  is a topology in  $X$ , then  $(X, \Gamma)$  is a space with topology  $\Gamma$ ; or shortly we call  $X$  a topological space. Each element in  $\Gamma$  is a open set in  $(X, \Gamma)$ .

**Definition 2.2** Let  $(X, \Gamma)$  be a topological space.  $K$  is a family of subsets of  $\Gamma$ . Supposing that each elements of  $\Gamma$  is union of some elements in  $K$ , or for all  $U \in \Gamma$ , there exists  $K_1 \in K$  such that  $U = \bigcup_{B \in K_1} B$ , then  $K$  is a basis of  $\Gamma$ , or we write  $K$  as a basis of  $X$ .

**Definition 2.3** Let  $(X, \Gamma)$  be a topological space,  $H$  be a family of subsets of  $\Gamma$ . Supposing that the union of all non-empty subsets in  $H$ , defined as

$$K = \{S_1 \cap S_2 \cap \dots \cap S_n \mid S_i \in H, i = 1, 2, \dots, n; n \in \mathbb{Z}_+\}$$

is a basis of  $\Gamma$ , then  $H$  is a sub-basis of  $\Gamma$ , or we write  $H$  as a sub-basis of  $X$ .

**Theorem 2.4** Let  $X$  be a set,  $K$  be a family of subsets of  $X$  ( $K \subset \Gamma(X)$ ). Supposing that  $\mathcal{B}$  satisfying,

- $\bigcup_{A \in K} A = X$
- $B_1, B_2 \in K$ , then for all  $x \in B_1 \cap B_2$ , there exists  $B \in K$  such that  $x \in B \subset B_1 \cap B_2$

the family of subsets of  $X$

$$\Gamma = \left\{ U \subset X \mid \exists K_U \subset K \Rightarrow U = \bigcup_{B \in K_U} B \right\}$$

is the unique topology of  $X$  which is basis of  $K$ ; vice versa, if a family of subsets  $K$  is a basis of topology in  $X$ , then  $K$  should satisfy (1) and (2).

**Definition 2.5** Let  $(X, \Gamma)$  be a topological space,  $x \in X$ . Supposing that  $U$  is a subset of  $X$  satisfying, there exists an open set  $V \in \Gamma$  such that  $x \in V \subset U$ , then  $U$  is a neighborhood of  $x$ . Therefore, if  $U$  is an open set with  $x$ , it should be a neighborhood of  $x$ . We call  $U$  be an open neighborhood of  $x$ .

**Definition 2.6** Let  $(X, \Gamma)$  be a topological space,  $x \in X$ . Supposing that  $\mu_x$  be a family of neighborhood of  $x$ . The subsets  $\nu_x$  of  $\mu_x$  satisfying, for all  $U \in \mu_x$ , there exists a set  $V \in \nu_x$  such that  $V \subset U$ . Then we define  $\nu_x$  as a basis of the family of neighborhoods of  $x$ , or shortly we write it as a neighborhood basis of  $x$ . If the subsets  $\omega_x$

satisfying, union of all non-empty finite subsets  $\omega_x$

1, Research partially supported by Scientific Research Funding granted by Capital University of Economics and Business No. 2014XJQ011.

2, Research partially supported by Teaching Reform Funding <Reform of courses <Real Variable Function and Functional Analysis> granted by Capital University of Economics and Business 2015.

$$\{W_1 \cap W_2 \cap \dots \cap W_n \mid W_i \in \omega_x, i=1,2,\dots,n; n \in \mathbb{Z}_+\}$$

is a neighborhood basis of  $\mu_x$ , then we define  $\omega_x$  be a sub-neighborhood basis of  $x$ .

### B. Orderly Sets

Definition 2.7 Let  $A$  be a set, ' $<$ ' be a relationship in  $A$  satisfying,

- transitivity: if  $x < y$  and  $y < z$ , then  $x < z$ .
- unreflexivity: there exists no  $x$  such that  $x < x$ .
- comparability: for all  $x$  and  $y$  satisfying  $x \neq y$ , either  $x < y$  or  $y < x$  holds.

We define ' $<$ ' as orderly relationship in  $A$ , and  $(A, <)$  as an orderly set.

Let  $(A, <)$  be an orderly set, for  $a < b$ , we write

$$(a, b) = \{x \mid a < x < b\}$$

$$(a, b] = \{x \mid a < x \leq b\}$$

$$[a, b) = \{x \mid a \leq x < b\}$$

$$[a, b] = \{x \mid a \leq x \leq b\}$$

$$[a, \infty) = \{x \mid a \leq x\}$$

$$(\infty, b] = \{x \mid x \leq b\}$$

Supposing that  $A_0$  is a subset of  $A$ . If  $b \in A_0$  and for all  $x \in A_0$ ,  $x \leq b$ , we define  $b$  as the maximum in  $A_0$ . Similarly, if  $a \in A_0$  and for all  $x \in A_0$ ,  $a \leq x$ ,  $a$  is defined as the minimum in  $A_0$ .

Example 2.8 Consider the relationship formed by all real pairs  $(x, y)$  satisfying  $x < y$ . That is a so call orderly relationship as defined above.

## III. ORDERLY TOPOLOGY

### A. Definition

Definition 3.1 Let  $X$  be a set with orderly relationship, containing more than one element. Supposing that  $K$  is a family of subsets covering:

- All subsets like  $(a, b)$  in  $X$ .
- All subsets like  $[a_0, b)$ , where  $a_0$  is minimum.
- All subsets like  $(a, b_0]$ , where  $b_0$  is maximum.

By Theorem 2.1  $K$  is a basis of some topology of  $X$ . Define that certain topology as an orderly topology and  $X$  as an orderly topological space.

Example 3.2 The standard topology of  $\mathbb{R}$  (built by all open set  $(a, b) = \{x \mid a < x < b\}$ ), is exactly an orderly topology by standard relationship in  $\mathbb{R}$ .

Example 3.3 :  $\mathbb{Z}_+$  is an orderly set with minimum. The orderly topology in  $\mathbb{Z}_+$  is a discrete topology, since each element is an open set.

### B. Left and right neighborhood(open set)

Definition 3.4 Let  $X$  be an orderly topological space,  $x \in U \subset X$  and  $x$  is not maximum. If there exists a neighborhood  $V \subset X$  of  $x$  such that  $U = V \cap [x, \infty)$ , we write  $U$  as the right neighborhood of  $x$ . All right neighborhood of  $x$  build a family of right neighborhood of  $x$ . Similarly, by replacing  $[x, \infty)$  with  $(-\infty, x]$ ,  $x$  not be minimum in  $X$ , we can define the left neighborhood of  $x$  and the family of left neighborhood.

Theorem 3.5 Let  $X$  be an orderly topological space,  $x \in X$ ,  $\mu_x^+$  is the family of right neighborhood of  $x$ ,

- If  $U \in \mu_x^+$ ,  $U \subset V$ , then  $V \in \mu_x^+$
- if  $U, V \in \mu_x^+$ , then  $U \cap V \in \mu_x^+$
- if  $v_x^+ \in \mu_x^+$ , then  $\bigcup_{U \in v_x^+} U \in \mu_x^+$

Proof:

(1) Obviously it is true.

(2) Since  $U, V \in \mu_x^+$ , there exist neighborhood  $P, Q$  of  $x$  such that

$$U = P \cap [x, \infty), V = Q \cap [x, \infty),$$

then

$$U \cap V = (P \cap Q) \cap [x, \infty)$$

where,  $P \cap Q$  is a neighborhood,  $U \cap V \in \mu_x^+$ .

(3) for all  $U \in v_x^+$ , there exist a neighborhood  $P$  of  $x$  such that  $U = P \cap [x, \infty)$ , then  $\bigcup_{U \in v_x^+} U = \bigcup (P \cap [x, \infty)) = P \cap [x, \infty)$ , which means  $\bigcup_{U \in v_x^+} U \in \mu_x^+$ .

Similarly, the theorem holds for the left neighborhood of  $x$ .

Definition 3.6 Let  $X$  be an orderly topological space,  $U \subset X$ . If for all  $x \in U \subset X$ , there exist  $U \cap [x, \infty)$  which is right neighborhood of  $x$ , we define  $U$  as right open set of  $X$ . All right open set build right topology of  $X$ . Similarly, we can define the left open set and left topology of  $X$ .

Theorem 3.7 Let  $X$  be an orderly topological space,  $\Gamma^+$  is right topology of  $X$ .

- If  $A$  is an open set in  $X$ , then  $A \in \Gamma^+$ .
- If  $A, B \in \Gamma^+$ , then  $A \cap B \in \Gamma^+$
- If  $\Gamma_1^+ \in \Gamma^+$ , then  $\bigcup_{U \in \Gamma_1^+} U \in \Gamma^+$

Proof:

(1) It is obviously true.

(2) For all  $x \in A \cap B$ , by  $A, B \in \Gamma^+$ ,  $A \cap [x, \infty), B \cap [x, \infty)$  should be right neighborhood of  $x$ , thus  $(A \cap B) \cap [x, \infty) = (A \cap [x, \infty)) \cap (B \cap [x, \infty))$

should be right neighborhood of  $x$ , therefore,  $A \cap B \in \Gamma^+$ .

(3) For all  $x \in \bigcup_{U \in \Gamma_1^+} U$ , there exists  $x \in U \in \Gamma_1^+ \subset \Gamma^+$ ,  $U \cap [x, \infty)$  should be right neighborhood of  $x$ ,

$$U \cap [x, \infty) \subset \left( \bigcup_{U \in \Gamma_1^+} U \right) \cap [x, \infty) = \bigcup_{U \in \Gamma_1^+} (U \cap [x, \infty))$$

Therefore,  $\bigcup_{U \in \Gamma_1^+} U \in \Gamma^+$ .

#### IV. LEFT AND RIGHT CONTINUITY IN ORDERLY TOPOLOGICAL SPACE

##### A. local and global left(right) continuity

**Definition 4.1** Let  $X$  be an orderly topological space,  $Y$  be a topological space and a map  $f: X \rightarrow Y$ . Assume that for each open set  $U$  in  $Y$ ,  $f^{-1}(U)$  is a right open set in  $X$ , we define  $f$  be a right continuous map from  $X$  to  $Y$ . Similarly, we can define a left continuous map.

**Theorem 4.2** Let  $X$  be an orderly topological space,  $Y$  be a topological space and a map  $f: X \rightarrow Y$ , then conclusions below are equal:

- $f$  is a right continuous map;
- there exist a basis  $K$  in  $Y$  such that for all  $U \in K$ , the preimage  $f^{-1}(U)$  is a right open set in  $X$ ;
- there exists a sub-basis  $H$  in  $Y$  such that for all  $S \in H$ , the preimage  $f^{-1}(S)$  is a right open set in  $X$ .

**Proof:** It is obvious that (1) yields (3), since the topology of  $Y$  is definitely a sub-basis of  $Y$ .

If (3) holds, assume  $H$  is a sub-basis of  $Y$ . By definition,

$$K = \{S_1 \cap S_2 \cap \dots \cap S_n \mid S_i \in H, i = 1, 2, \dots, n; n \in \mathbb{Z}_+\}$$

is a topological basis of  $Y$ . For all  $S_i \in \Gamma, i = 1, 2, \dots, n$ , where  $n \in \mathbb{Z}_+$ , we get

$f^{-1}(S_1 \cap S_2 \cap \dots \cap S_n) = f^{-1}(S_1) \cap f^{-1}(S_2) \cap \dots \cap f^{-1}(S_n)$ , which is union of right open set in  $X$ , thus it is a right open set in  $X$ .

If (2) holds, assume  $\Gamma$  is a basis of  $Y$ . If  $U$  is an open set in  $Y$ , there exists  $K_1 \in K$  such that

$$U = \bigcup_{B \in K_1} B, \quad \text{then we get}$$

$f^{-1}(U) = f^{-1}\left(\bigcup_{B \in K_1} B\right) = \bigcup_{B \in K_1} f^{-1}(B)$  as union of a family of right open set in  $X$ , thus it is a right open set in  $X$ . That indicates that  $f$  is a right continuous map.

Theorem 4.2 holds for the left continuous map.

**Definition 4.3** Let  $X$  be an orderly topological space,  $Y$  be a topological space and a map  $f: X \rightarrow Y, x \in X$ , with  $x$  not the maximum of  $X$ . For all preimage  $f^{-1}(U), f^{-1}(U) \cap [x, \infty)$  is a right neighborhood of  $x \in X$ , we define  $f$  be a right continuous map at  $x$ . Similarly, we can define a left continuous map at  $x$ .

**Theorem 4.4** Let  $X$  be an orderly topological space,  $Y$  be a topological space and a map  $f: X \rightarrow Y$ , the conclusions below are equal:

- $f$  is a continuous map at  $x$ ;
- there exists a basis of neighborhood  $\nu_{f(x)}$  for  $f(x)$  such that for all  $V \in \nu_{f(x)}$ , the preimage  $f^{-1}(V) \cap [x, \infty)$  is a right neighborhood of  $x$ .
- there exists a sub-basis of neighborhood  $\omega_{f(x)}$  for  $f(x)$  such that for all  $W \in \omega_{f(x)}$ , the preimage  $f^{-1}(W) \cap [x, \infty)$  is a right neighborhood of  $x$ .

**Proof:** It is obvious that (1) yields (3) since the family of neighborhood of  $f(x)$  is definitely a sub-basis of neighborhood of  $f(x)$ .

If (3) holds, assume that  $\omega_{f(x)}$  is a sub-basis of neighborhood for  $f(x)$ . By definition, the family of subsets  $\{W_1 \cap W_2 \cap \dots \cap W_n \mid W_i \in \omega_x, i = 1, 2, \dots, n; n \in \mathbb{Z}_+\}$  is a basis of neighborhood for  $f(x)$ . For all  $W_i \in \omega_x, i = 1, 2, \dots, n$ , where  $n \in \mathbb{Z}_+$ , we get  $f^{-1}(\cap W_i) \cap [x, \infty) = \cap f^{-1}(W_i) \cap [x, \infty)$ , which is union of  $n$  right neighborhood of  $x$ , thus it is a right neighborhood of  $x$ . That comes the conclusion (2).

If (2) holds, assume that  $\nu_{f(x)}$  is a basis of neighborhood for  $f(x)$ . If  $U$  is a neighborhood of  $f(x)$ , there exists  $V \in \nu_{f(x)}$  such that  $V \subset U$ , thus  $f^{-1}(V) \subset f^{-1}(U)$ . Noticing that  $f^{-1}(V) \cap [x, \infty)$  is a right neighborhood of  $x$ ,  $f^{-1}(U) \cap [x, \infty)$  is also a right neighborhood of  $x$ , which indicates that  $f$  is a continuous map at  $x$ .

Similarly, this theorem holds still for left continuous map at  $x$ .

##### B. Connections between global and local continuous maps

**Theorem 4.5** Let  $X$  be an orderly topological space,  $Y$  be a topological space and a map  $f: X \rightarrow Y$ .  $f$  is right continuous if and only if for all  $x \in X$ ,  $f$  is right continuous at  $x$ .

**Proof:** Necessity. Assume that  $f$  is right continuous,  $x \in X$ . If  $U$  is a neighborhood, there exists  $V$  such that  $f(x) \in V \subset U$ , then  $x \in f^{-1}(V) \subset f^{-1}(U)$ , where  $f^{-1}(V)$  is a right open set. Therefore,  $f^{-1}(V) \cap [x, \infty)$  is a right neighborhood for  $x \in X$ , which

yields that  $f^{-1}(U) \cap [x, \infty)$  is a right neighborhood for  $x \in X$ .  $f$  is right continuous at  $x$ .

Sufficiency. For all  $x \in X$ ,  $f$  is right continuous at  $x$ . If  $U \subset Y$  is an open set, then for all  $x \in f^{-1}(U)$ ,  $U$  is a neighborhood. Therefore, for all  $x \in f^{-1}(U)$ ,  $f^{-1}(U) \cap [x, \infty)$  is a right neighborhood for  $x$ , then  $f^{-1}(U)$  is a right open set.

This theorem holds still for left continuous map.

**Theorem 4.6** Let  $X$  be an orderly topological space,  $Y$  be a topological space and a map  $f: X \rightarrow Y$ .  $f$  is continuous if and only if  $f$  is left and right continuous.

**Proof:** Necessity.  $f$  is continuous, then for open set  $U$  in  $Y$ ,  $f^{-1}(U)$  is an open set in  $X$ . It is obvious that  $f^{-1}(U)$  is left and right open set in  $X$ , which means that  $f$  is left and right continuous.

Sufficiency. Assume that  $f$  is left and right continuous, for open set  $U$  in  $Y$ ,  $f^{-1}(U)$  is left and right open set in  $X$ . That is for any  $x \in f^{-1}(U)$ , there exist neighborhood  $P, Q$  of  $x$  such that  $f^{-1}(U) \cap [x, \infty) = P \cap [x, \infty)$ ,  $f^{-1}(U) \cap (-\infty, x] = Q \cap (-\infty, x]$ .

Thus

$f^{-1}(U) = (f^{-1}(U) \cap [x, \infty)) \cup (f^{-1}(U) \cap (-\infty, x]) = (P \cap [x, \infty)) \cup (Q \cap (-\infty, x])$  is neighborhood for  $x$ , which means that  $f^{-1}(U)$  is an open set.

**Example 4.7** The left(right) map in orderly topological space is not always a continuous map.

$\mathbb{R}$  is standard topological space and define  $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

It is obvious that  $f(x)$  is right continuous at 0 but not continuous.

#### ACKNOWLEDGEMENT

The author would like to thank all the editors.

#### REFERENCES

- [1] Xiong, Jincheng, Lectures on general topology, 3rd edition, Higher Education Press, 2003.
- [2] James R. Munkres, Topology, 2nd edition, Mechanic and Industry Press, 2006.
- [3] John D. Baum, Principles on general topology, People's Education Press, 1981.
- [4] Li, Panlin, Li, Lihan, Li, Yang, Wang, Chunli, Discrete Mathematics, Higher Education Press, 1999.
- [5] Chen, Jixiu, Yu, Chonghua, Jin, Lu, Mathematical Analysis, Higher Education Press, 1999.
- [6] Guo, Dajun, Huang, Chunchao, Liang, Fanghao, Wei, Zhongli, Real Variable Function and Functional Analysis, Shandong Univ. Press, 2005.