To Obtain the OWA Weighting Vector Via Normal Distribution Function

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ABSTRACT: The OWA operator is an important assessment method in multiple-attribute decision making. A new model based on the density function of normal distribution is given to assign reasonable weights of OWA operators. Using the orness level as a parameter in this method, one can deduce OWA weights by solving a quadratic programing problem. Three propositions of this new model are proven, and a numerical example about the assessment of red wines is given to analyze and illustrate this method.

KEYWORD: OWA operators; normal distribution; weighting vector

1 INTRODUCTION

The ordered weighted averaging (OWA) operator proposed by Yaer [1] is an important averaging method in multiple-attribute decision making areas such as effectiveness evaluation, neural networks, fuzzy logic controllers [2][3]. Weighting vector is one of the most important issues in an OWA assessment, and the value of a weighting vector is often affected by the degree of optimism of an evaluator.

Though some simple weighing vectors, such as maximum weights; minimum weights and trimmed weights have many practical usages in real life, they cannot meet an evaluator's need when he want to emphasizes the importance of any specific median value; and they cannot be adjusted with the change of the evaluator's degree of optimism. Some model-based solution partially solve the problems mentioned above. For example, the maximum entropy [4][5], the minimum variability [6], least square deviation [7], minimax disparity approach [8][9], and so on [10].

In [11], we proposed a model based on combination number to obtain a kind of weighting vector. In this paper, a similar model to obtain the OWA weights based on the normal distribution function is proposed.

The following paper is organized as follow. In section 2, the OWA operators and some models of OWA models are introduced. In section 3, a new model is given, and 3 properties of this model are proofed. And finally in section 4, a numerical example about wine assessment is given to show that our new model can reflect the degree of optimism fairly well.

2 OWA OPERATORS AND MODELS

2.1 *OWA Operators*

A n dimensional OWA operator with a weighting vector $W = (w_1, w_2, ..., w_n)$, $w_1 + w_2 + \cdots + w_n = 1$, $0 \le w_i \le 1$, is a function $F_W : R^n \rightarrow R$, such that[1]

$$F_W(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} w_i y_i , \qquad (1)$$

where y_i is the i-th largest element in $(x_1, \cdots, x_n)$.

For a weighting vector $W = (w_1, w_2, ..., w_n)$, the degree of orness level α is defined as:

$$\text{orness}(W) = \alpha(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i, \qquad (2)$$

It is obvious that $\alpha \in [0,1]$. And the degree of optimism of an evaluator can be expressed by the orness level. For example, for a weighting vector $W_{\max} = (1, 0, ..., 0)$, where an optimistic evaluator consider only the best value in all the attributes, $\alpha(W_{\max}) = 1$. And for another weighting vector $W_{\min} = (0, ..., 0, 1)$, where the worst value is the only one that a pessimistic evaluator cares, $\alpha(W_{\min}) = 0$. When the evaluator prefers the mean

of $(x_1, \cdots, x_n)$, where $W_{mean} = (1/n, 1/n, \ldots, 1/n)$ and every value is considered equally important, $\alpha(W_{mean}) = 1/2$.

2.2 Models to obtain weighting vector

Besides W_{max}, W_{min} and W_{mean} mentioned above, trimmed mean is another weighting vector that is frequently used. For example, it is used in the assessment of diving competition in 2012 London Olympic Games. Regardless of the largest m numbers and smallest m numbers, trimmed mean averages the rest of the values, thus corresponds to a weighting vector like

$$W_{trim} = \left(0, \cdots, 0, \frac{1}{n-2m}, \cdots, \frac{1}{n-2m}, 0, \cdots, 0\right)$$

Different scholars proposed different models to generate weighting vectors. For example, the least square deviation model [7]:

$$\begin{cases} \min = \sum_{i=1}^{n-1}(w_i - w_{i-1})^2 \\ s.t. \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha \\ \sum_{i=1}^{n} w_i = 1 \\ w_i \geq 0, \ i = 1, 2, \cdots, n \end{cases}$$

and combination number based OWA model [11]:

$$\begin{cases} \min = \sum_{i=1}^{n}\left(w_i - \frac{C_{n-1}^{i-1}}{2^{n-1}}\right)^2, \\ s.t. \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \\ \sum_{i=1}^{n} w_i = 1, \\ w_i \geq 0, i = 1, 2, \cdots, n, \end{cases}$$

3 A NEW MODEL TO OBTAIN OWA WEIGHTS

In this section, we propose a new model to obtain the OWA weights base on normal distribution function as follow:

$$\begin{cases} \min J_1 = \sum_{i=1}^{n}(w_i - \tilde{w}_i)^2, \\ s.t. \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \\ \sum_{i=1}^{n} w_i = 1, \\ w_i \geq 0, i = 1, 2, \cdots, n, \end{cases} \tag{3}$$

where $\tilde{w}_i = \dfrac{\Phi(a_i) - \Phi(a_{i-1})}{\Phi(a_n) - \Phi(a_1)}$, $a_i = i - \dfrac{n}{2}$.

It is easy to prove that $\tilde{w}_i$ is strictly monotone increasing in $[1, n/2]$, and decreasing in $[n/2, n]$; and $\tilde{w}_i$ is symmetrical to the most center element of $(\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n)$, that is $\tilde{w}_i = \tilde{w}_{n-i}$. Such properties ensure that values in the middle play more important roles than values in the two ends, which is similar to trimmed mean. Given $n = 5$, the solution of (3) is shown in Figure 1, from which one can see that the obtained weights are symmetric, and they can reasonably adjusted by the orness level.
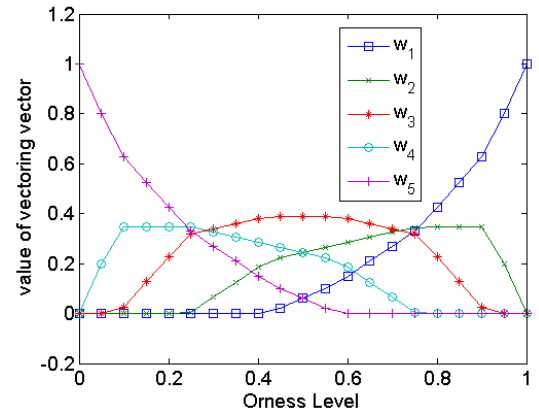

Fig.1. Result of the new model

Below we will prove some properties of the new proposed OWA model:

Theorem1. $W_{max} = (1, 0, \ldots, 0)$, $W_{min} = (0, \ldots, 0, 1)$ are the optimal solutions of model (3) if and only if orness level $\alpha = 1$ and $\alpha = 0$.

Proof: (Sufficient conditions)

when $W_{max} = (1, 0, \ldots, 0)$,

$$\alpha = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \frac{n-1}{n-1} \times 1 + 0 + \cdots + 0 = 1.$$

(**Necessary conditions**) Notice $\sum_{i=1}^{n} w_i = 1$, we have

$$\alpha = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \sum_{i=1}^{n}\left(1 - \frac{i-1}{n-1}\right) w_i$$

$$= \sum_{i=1}^{n} w_i - \sum_{i=1}^{n} \frac{i-1}{n-1} w_i = 1 - \sum_{i=1}^{n} \frac{i-1}{n-1} w_i = 1$$

then

$$\sum_{i=1}^{n} \frac{i-1}{n-1} w_i = \frac{1}{n-1} w_2 + \frac{2}{n-1} w_3 + \cdots + \frac{n-2}{n-1} w_{n-1} = 0$$

since $w_i \geq 0$, we have

$$w_1 = 1, w_2 = w_3 = \cdots = w_n = 0.$$

Thus we prove $W_{\max} = (1,0,...,0)$ is the optimal solution if and only if $\alpha = 1$. Similarly, we can prove that $W_{\min} = (0,...,0,1)$ is the optimal solution if and only if $\alpha = 0$.

Theorem2. $\tilde{W} = (\tilde{w}_1, \tilde{w}_2,...,\tilde{w}_n)$ is the optimal solutions of model (3) if and only if orness level $\alpha = 1/2$.

Proof：(Sufficient conditions) if $n = 2m$ is even, then

$$\alpha = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \sum_{i=1}^{2m} \frac{n-i}{n-1} \tilde{w}_i$$
$$= \sum_{i=1}^{m} \left(\frac{n-i}{n-1} + \frac{n-1-(n-i)}{n-1} \right) \tilde{w}_i .$$
$$= \sum_{i=1}^{m} \tilde{w}_i = \frac{1}{2}$$

If $n = 2m+1$ is odd, then

$$\alpha = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \sum_{i=1}^{2m+1} \frac{n-i}{n-1} \tilde{w}_i$$
$$= \sum_{i=1}^{m} \left(\frac{n-i}{n-1} + \frac{n-1-(n-i)}{n-1} \right) \tilde{w}_i + \frac{n-m-1}{n-1} \tilde{w}_{m+1}$$
$$= \sum_{i=1}^{m} \tilde{w}_i + \frac{1}{2} \tilde{w}_{m+1} = \frac{1}{2}$$

(**Necessary conditions**) If the objective function $J_1 = \sum_{i=1}^{n} (w_i - \tilde{w}_i)^2$ of model (3) achieves the minimum zero, then w_i must equal to $\tilde{w}_i$. Therefore, the weighting vector W is equal to $\tilde{W}$. So, $\tilde{W}$ is the optimal solution of model (3).

Theorem3. If $W = (w_1, w_2, \cdots, w_n)$ is the optimal solutions of model (3) when the value of orness level is α, then $w = (w_n, w_{n-1}, \cdots, w_1)$ if the optimal solution of (3) when the value of orness level is $1-\alpha$.

Proof: Under a orness level of α, if $W = (w_1, w_2, \cdots, w_n)$ is the optimal solution of (3), then its objective function $J_1 = \sum_{i=1}^{n} (w_i - \tilde{w}_i)^2$ achieves the minimum, and its conditions

$\sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha$, $\sum_{i=1}^{n} w_i = 1$, holds. Now we will show that $w = (w_n, w_{n-1}, \cdots, w_1)$ is the optimal solution of

$$\begin{cases} \min J_2 = \sum_{i=1}^{n} (w_i - \tilde{w}_i)^2, \\ \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = 1-\alpha, \\ \sum_{i=1}^{n} w_i = 1, \ w_i \geq 0, i = 1, 2, \cdots, n, \end{cases}$$

For one thing, since $\sum_{i=1}^{n} w_{n+1-i} = 1$

and $\sum_{i=1}^{n} \frac{n-i}{n-1} w_{n+1-i} = 1 - \sum_{i=1}^{n} \frac{i-1}{n-1} w_{n+1-i} = 1-\alpha$

$w = (w_n, w_{n-1}, \cdots, w_1)$ is a feasible solution. For another, because $\tilde{w}_i = \tilde{w}_{n-i}$, $J_2 = \sum_{i=1}^{n} (w_{n+1-i} - \tilde{w}_i)^2$ is always equal to J_1, the objective function of (3). Since J_1 achieves minimum at $(w_1, w_2, \cdots, w_n)$, J_2 achieves minimum at $(w_n, w_{n-1}, \cdots, w_1)$. That is to say, $(w_n, w_{n-1}, \cdots, w_1)$ is the optimal solution of the model with an orness level of $1-\alpha$.

4 EXPERIMENT

27 red wines are to be assessed by a wine commission that consists of 10 professional tasters. Each taster is require to assess appearance, color, flavor, taste, and overall impression of each wine. Total scores of these assessment are listed in Table 1.

Tab.1. scores of 22 red wines from 10 judgers

	J1	J2	J3	J4	J5	J6	J7	J8	J9	J10
wine1	71	81	86	74	91	80	83	79	85	73
wine2	80	85	89	76	69	89	73	83	84	76
wine3	52	64	65	66	58	82	76	63	83	77
wine4	74	74	72	62	84	63	68	84	81	71
wine5	72	69	71	61	82	69	69	64	81	84
wine6	63	70	76	64	59	84	72	59	84	84
wine7	64	76	65	65	76	72	69	85	75	76
wine8	77	78	76	82	85	90	76	92	80	79
wine9	67	82	83	68	75	73	75	68	76	75
wine10	73	60	72	63	63	71	70	66	90	73
wine11	69	84	79	59	73	77	77	76	75	77
wine12	70	77	70	70	80	59	76	76	76	76
wine13	72	80	80	71	69	71	80	74	78	74
wine14	70	79	91	68	97	82	69	80	81	76
wine15	76	84	84	66	68	87	80	78	82	81
wine16	78	84	76	71	82	79	76	76	86	81
wine17	73	90	96	71	69	60	79	73	86	74
wine18	73	83	72	68	93	72	75	77	79	80
wine19	70	85	90	68	90	84	70	75	78	70
wine20	60	78	81	62	70	67	64	62	81	67
wine21	73	80	71	61	78	71	72	76	79	77
wine22	70	77	63	64	80	76	73	67	85	75

Set α be 0.1, 0.3, 0.5, 0.7 and 0.9 respectively, and use model (3) to calculate the weighting vectors. The OWA assessments of 27 wines under different orness levels are listed in Table 2, from which one can see that for an optimistic evaluator(when α is 0.7 or 0.9), No.14 is viewed as the best, because the highest score, 97, from No.5 judger plays an important role in the assessment for an optimistic evaluator. No.2 wine is reviewed as the best in a neutral evaluator's eyes(when α is 0.5), because it receives fairly good scores from most of the judgers. Though No.8 wine receives no scores higher than 90 from the 10 judgers, it is still regarded as the best wine for a pessimistic evaluator(when α is 0.1 or 0.3), because each judger give it a not-too-bad score, which is higher than 76.

Tab.2. OWA assessments under different orness levels

α	0.9	0.7	0.5	0.3	0.1
wine1	87.87	84.06	80.59	76.84	72.83
wine2	87.95	84.71	81.05*	76.60	71.97
wine3	80.69	73.52	66.93	61.36**	56.38**
wine4	82.62	77.52	72.93	68.73	64.00
wine5	81.77	75.63	70.36	67.12	63.67
wine6	82.95	77.11	70.84	65.38	60.34
wine7	80.29	76.77	73.09	69.09	65.10
wine8	89.43	84.21	79.71	77.90*	76.38*
wine9	81.04	77.62	74.60	71.63	68.14
wine10	81.06	75.18	69.98	66.17	62.09
wine11	80.97	78.48	76.29	71.48	65.32
wine12	78.23**	76.94	75.05	70.53	65.03
wine13	79.56	77.01	74.34	72.10	70.25
wine14	91.73*	84.97*	79.08	74.53	69.53
wine15	85.19	82.78	80.33	75.21	69.14
wine16	84.30	81.38	78.53	76.10	73.70
wine17	91.00	81.97	74.41	69.89	65.30
wine18	86.95	81.12	76.00	73.08	70.28
wine19	88.43	82.76	76.69	72.46	69.32
wine20	79.34	72.96	67.14**	64.14	61.41
wine21	79.10	76.91	74.50	70.61	66.32
wine22	81.59	77.58	73.64	69.31	64.67

* maximum score
** minimum score

5 CONCLUSIONS

Based on the good properties of normal distribution function, a new method to generate a weighting vector for OWA assessment is proposed. Our model is a quadratic programming problem. Similar to the well-known trimmed mean, such weighting vector emphasizes the values in the middle place, and three properties about such weighting vector, which is the solution of our model, is proofed. These properties

show that the parameter of our model, namely, the orness level, can represent the degree of optimism of an evaluator. As an example, we use these weighting vectors to assess the qualities of different wines, under different orness levels. The result of this example is consistent with subjective sensations of different people with different degrees of optimism.

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