The Image Thinking and the Abstract Thinking in "Linear Algebra" Course Teaching

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ABSTRACT: In the process of "linear algebra" teaching, we should combine with application of image thinking and abstract thinking. This paper illustrates the iconicity, the non-logical, the rough and the imaginative features of image thinking and the analysis and synthesis, the classification and comparison, the abstraction and generalization, the induction and deduction in logical thinking.
KEYWORD: Linear algebra; Image thinking; Abstract thinking

1 INTRODUCTION

Modern mathematics teaching theory shows that mathematics is a process of thinking activity and mathematics teaching is the teaching of math thinking activities [1]. We must take the thinking activities in mathematics teaching as the main objects of teaching and research. Thinking activities in mathematics teaching mainly have the following three contents: the mathematician thinking activity, the mathematical teachers' thinking activity and the students' thinking activity [2]. Therefore, we should focus on displaying mathematical thinking process, and revealing the thinking process of mathematicians through our thinking process, and inducing the thinking process of students. Those are the guarantees of successful mathematics teaching activities and forming good teaching structure.

As a teacher, we most widely apply and most effective thinking are the image thinking and abstract thinking in teaching activities, and even consider the image thinking and abstract thinking together. Also, we should deal with the teaching of algorithm and the training of thinking properly, and arrange the teaching time reasonably. In the long-term teaching practice, we found that the students master the algorithm generally good, and do some simple image thinking as well. However, most students feel that it is difficult to deal with some more complex image thinking and abstract thinking, some students even have conflicted mood. In view of this, the discussion of the training methods of these two thinking activities is very necessary.

2 THE IMAGE THINKING IN "LINEAR ALGEBRA" COURSE TEACHING

The image thinking is mainly formed when people choose or give up some appearance in the process of learning about the world, which is a thinking method that only applying the visual image representation to solve some problems. According to combine the subjective cognition and emotion, and apply a certain form and some means and tools to create and describe the image, image thinking can feel and storage the objective image system. In the process of "linear algebra" course, the image thinking has the iconicity, the non-logicality, the roughness and the imagination features.

Iconicity is the most basic feature of image thinking. The reflected object of image thinking is the image of the things. The thinking form of image thinking is the notion of iconicity, such as images, audio-visual, imagine tools. The expression means are graphics, images, schema and the vivid symbols, which all can be identified by sensory perception. For example, when introduce the third order determinant expansion formula, we can draw a graphic with six lines, including three lines for the sign "+", the other three lines for the sign "-". The three elements in the same line are multiplied, so that students will soon be able to remember the six expansion formula.

The non-logicality is also a feature of image thinking. The information processing of image thinking is not serial but parallel, which is surface or solid. It can make the thinking subject quickly seize the problem as a whole. Such as when introduce the six major properties of determinant, we usually explain one by one firstly, and then apply all of the properties in a typical calculation of determinant. If so, the students can quickly from the overall grasp for the calculation of the determinant.

The reflection of image thinking is the rough line reflection, which grasps the question is in general and analyzes the problem is qualitative or half quantitative. So, image thinking is typically used for qualitative analysis of the problem. Generally, we often need combine abstract thinking with image thinking in the actual thinking activity. For example, when introduce the maximum impertinent groups of vector group, we first arrange all vectors in a matrix, then transform this matrix to the ladder type by elementary line transformation. We can directly find the maximum impertinent groups from the rows of non-zero line. However, if we want to linear express the rest of the vectors by the maximum impertinent groups, which involves accurate quantitative relation and need apply the abstract thinking to do further analysis.

The imagination is a typical feature of image thinking. Imagination is the process that the thinking subject uses the existing image to form a new image. Image thinking not satisfies with the reproduction of the existing image, and more committees to process the existing image in order to get an output of new product. For example, when introduce Schmidt orthogonalization process, $\beta_1=\alpha_1$, $\beta_2=\alpha_2-\frac{[\beta_1,\alpha_2]}{[\beta_1,\beta_1]}\beta_1$, $\beta_3=\alpha_3-\frac{[\beta_1,\alpha_3]}{[\beta_1,\beta_1]}\beta_1-\frac{[\beta_2,\alpha_3]}{[\beta_2,\beta_2]}\beta_2$, we can imagine β_2 as α_2 minus its projection in β_1, and imagine β_3 as α_3 minus its projection in β_1 also minus its projection in β_2. And so on, we can get the calculation formula for β_n.

3 THE ABSTRACT THINKING IN “LINEAR ALGEBRA” COURSE TEACHING

Abstract thinking is the process that people use concept, judgment, reasoning and so on to make indirect and general response for objective reality, which belong to rational knowledge stage [3]. According to abstract concept, abstract thinking reflects the nature of things and profound process of the development of the objective world, and makes people get knowledge through understanding activities, which goes far beyond by direct perception. Scientific abstraction is a reflection for nature by concept, or the intrinsic natural thought of the social material process. Based on observation, comparison, analysis, synthesis, abstraction, generalization, judgment and reasoning for the essential attribute of things, we extract the essential attribute of things, and make the knowledge from the perceptual concrete to abstract provision and form a concept eventually.

Abstract thinking ability is not only the essential ability to learn math well, but also the ability to learn other subjects and to deal with daily life. In the process of "“linear algebra” " teaching, the most effective method of training the abstract thinking ability is the ways that teacher poses questions and the students think for themselves or discuss each other to get the answers [4]. Specifically, abstract thinking have four thinking process, including analysis and synthesis, classification and comparison, abstraction and generalization, inductive and deductive. In the following, we take "“linear algebra” " teaching process for examples.

3.1 *Analysis and synthesis*

Analysis is the logical method in thinking which decomposes the object or factors into parts and researches respectively. Synthesis is the logical method in thinking which combines all part of the object or factors to become a unity and then give researches. We take the proof for the rule of expanding determinant by low in [5] as an example. Consider a general n-order determinant $D=\left(a_{ij}\right)_{n\times n}$, and apply analysis method, we decompose this determinant as the following n special determinant:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & & \\ 0 & a_{i2} & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}, \cdots,$$

$$\begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \\ 0 & \cdots & a_{ij} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}, \cdots, \begin{vmatrix} a_{11} & \cdots & a_{1,n-1} & a_{1n} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & a_{in} \\ \vdots & & \vdots & \\ a_{n1} & \cdots & a_{n,n-1} & a_{nn} \end{vmatrix}$$

Consider the values of these special determinants one by one. According to exchange the elements in matrix row by row and column by column, we can exchange the element a_{ij} to the position of the first row and the first column. Moreover, the rest of the elements all keep the original positions except the elements of the i-th row. Thus, we can get the values of these determinants are $D_1=a_{i1}A_{i1}$, $D_2=a_{i2}A_{i2},\ldots,$ $D_j=a_{ij}A_{ij},\ldots,$ $D_j=a_{ij}A_{ij},\ldots,$ $D_n=a_{in}A_{in}$. Finally, we raise the question: comprehensive consider the sum of these n determinants, what conclusion can we get? Automatically, we can get the conclusion $D_1+\cdots+D_n=\mathrm{D}$, which just the expansion formula by the i-th row of the determinant $D=a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots++a_{ij}A_{ij}+\cdots+a_{in}A_{in}$.

3.2 *Classification and comparison*

According to the intercommunity and differences, we can classify things. Comparison is to compare two or two kinds of things in common and differences. We take the presentation of the inverse matrix concept as an example.

As we all know, addition and subtraction and multiplication and division are the four operations for real number. For matrix, there are three operations, which are addition and subtraction and multiplication. But there is only no division operation. For the division operation of real number, there is $a \div b = a \times \frac{1}{b}$ $(b \neq 0)$. (If $b = 0$, then $a \div b$ is meaningless). Accordingly, we raise the question: Is there similar concept to the concept of reciprocal in matrix operating? There is a special number "1" in real number operating. When $a \neq 0$, there is $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$. There is a special matrix E in matrixes, which is similar to the number "1" in real numbers. Automatically, we propose the define of inverse matrix, which is "if $AB = BA = E$, then $B = A^{-1}$ or $A = B^{-1}$". Consider the necessary and sufficient conditions of the existing inverse matrix of matrix A continually, and make the determinant operation for the two sides of the define equation, i.e. $|AA^{-1}| = |A||A^{-1}| = |E| = 1$, we can get the necessary condition is $|A| \neq 0$. The sufficient condition can be reduced by the formula $AA^{*} = A^{*}A = |A|E$. If the matrix is inverse, and there need to divide to a matrix formally, we can do the multiplication which multiple the inverse matrix. For example, when solve the linear equations with its coefficient matrix is a square matrix $AX = B$, we can get $X = A^{-1}B$ when $|A| \neq 0$. But here we have to compare the matrix multiple inverse matrix with real number multiple reciprocal. For example, we compare the equation $a \div b = a \times \frac{1}{b} = \frac{1}{b} \times a \ \ (b \neq 0)$ with $X = A^{-1}B \neq BA^{-1}$. The difference of these two operations is that the multiplication of real numbers can exchange the orders but the multiplication of matrix not always can exchange the orders of operations. The formal division of matrix is really the multiplication essentially. Such as the former example, which essence is left multiple A^{-1} for the two sides of the equation. Here we must pay attention to that if left multiplication whether right multiplication.

3.3 *Abstraction and generalization*

Abstraction is extracting the essential attribute of the objects and abandoning the other no-essential attributes by using the power of the thinking [6]. Generalization is the way of thinking that generalizing a separate attribute of an object to the whole of this kind thing. We take solving the linear equations applying the elementary transformation as an example.

We apply the following three transformations for the linear equations in solving linear equations by the method of elimination. There first is exchanging the position of two equations. The second is multiplying a non-zero real number for an equation. And the third is adding one of the equations to another equation after multiplying a non-zero real number. Applying these three transformations, we can get new linear equations which have the same solutions with the original linear equations. After make abstract analysis for the former proceeding, we find that the essence of the method of elimination is make the coefficient of some unknown variable element in some equation to be zero. Fix the orders of the unknown variable elements, and suppose the first to the fourth columns representing the unknown numbers x_1, x_2, x_3, x_4, respectively, and suppose the fifth column representing the constant number item b, we find that the above proceedings only concerns the coefficients and constant number, and the known numbers are not concerned in the operations.

According, we can construct an augmented matrix including the coefficients and constant number of the equations, and make the respective elementary low transformation for this augmented matrix. Take the elementary low transformation continually until the augmented matrix is low standard simplest form matrix, we can get the general solutions of the equations. Summarizing the above proceedings, we can apply the elementary low transformation of matrix to the linear equations with n unknown varies and m equations. Meanwhile, we propose the one problem: "Whether we can apply the elementary transformation of matrix to solve the problems?" As long as the students grasp the essence of this method, they can easily get the right answer, which is that the former $n-1$ columns can exchange but the known vary of each column must follow the transformation, and the last column can not exchange with the former columns. There are also the other problem: "what's the role the elementary column transformation of matrix play for the different general solutions?" The students consider sequentially. In the low ladder type matrix, the first non-zero columns of the non-zero lows represent the non-freedom unknown varies. When we apply the elementary low transformations to solve the linear equations, we have the trend that selecting the former non-known varies as the non-freedom varies as far as possible. Applying the elementary column transformation of matrix may change the selections of the non-freedom varies, so we can get the general solutions with different forms.

3.4 *Induction and deduction*

Induction is from the precondition of individuality to general conclusion, and the relation of precondition and conclusion is probable **[7]**. We take the construction of the non-homogeneous linear equations $Ax = b$ as an example. There exist a question: "Whether we can get the general solutions of $Ax = b$ from the general solutions of the homogeneous linear equations $Ax = 0$?" If the general solutions of $Ax = 0$ are $x = k_1\xi_1 + k_2\xi_2 + \cdots + k_{n-r}\xi_{n-r}$, and a random solution of $Ax = b$ is η, then $\eta - \eta^*$ must be the solution of $Ax = 0$. We can get that $\eta - \eta^* = k_1\xi_1 + k_2\xi_2 + \cdots + k_{n-r}\xi_{n-r}$, i.e. $\eta = k_1\xi_1 + k_2\xi_2 + \cdots + k_{n-r}\xi_{n-r} + \eta^*$. Accordingly, we get the construction of the non-homogeneous linear equations $Ax = b$ is that the general solution of non-homogeneous linear equations equal to the general solution of homogeneous linear equations plus a random solution of non-homogeneous linear equations.

Deduction is from the precondition of generality to individual conclusion, and the relation of precondition and conclusion is necessary. We take the diagonalization for matrix and iagonalization for symmetric matrix as an example. As we all known, the sufficient and necessary condition for a n-order matrix A can be diagonalized is that there existing n linear independent eigenvectors which can construct a inverse matrix $P = (p_1, p_2, \cdots, p_n)$, such that $P^{-1}AP = \Lambda$. This is a general conclusion for a random n-order matrix. We propose the question: "What is the more special conclusion for the diagonalization of a symmetric matrix?" Because the eigenvalues of the symmetric matrix are all real numbers, and the eigenvectors of different eigenvalues are orthogonal, we can get a more special conclusion. That is the symmetric matrix can certainly be diagonalized, and there existing matrix P not only inverse but also orthogonal, such that $P^{-1}AP = P^T AP = \Lambda$.

4 CONCLUSIONS

In a word, in the teaching of the linear algebra course, we should strengthen to initiate the mathematical basic thinking and method and to train students' scientific thinking methods. We need synthesize application the image thinking and abstract thinking. Only in this way, we could receive good effect in practical teaching and enhance the students' overall thinking abilities.

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