# Reliability of a repairable system

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**Abstract.** In this note, we study the reliability a repairable system by using the result of the positive unit eigenfunction of the eigenvalue 0 of the system operator is just steady-state solution of the system. We give the stationary availability and the mean up-time etc., of the complex standby system. Moreover, numerical results are provided to investigate the effects of various system parameters on the reliability indices.

## Introduction

With the development of the modern technology and the world economy, the reliability problem of a system attracts more and more attentions. More and more people are working on the reliability of various systems [1]-[5]. In the same time, many valid methods have been developed to deal with the reliability for different systems. The repairable system is one of basic subjects in reliability. The availability and mean up-time are good evaluations of a repairable system performance, and occupy an increasingly important issues in power plant, manufacturing system and standby system. There is extensive literature on availability characteristics of repairable systems under various assumptions on the failures and repairs, see [6] and the references therein.

In most of these articles, methods used in the existing literature dealing with non-Markov systems involving many general random variables include the regenerative point technique (RPT) [6] and the supplementary variables method (SVM) [8]-[11]. In order to use the RPT, one has to correctly formulate and solve a system of Markov renewal equations, usually using an analytical method which is difficult for a non-Markov repairable system with only a few renewal points. By using the SVM, one can readily obtain all differential equations in terms of the state transition diagram of the model. However, it is still not easy to solve these differential equations because they usually involve some functions to be determined if there are at least two hazard rate parameters in one of the equations.

To overcome the above obstacles, we introduce a new method to calculate the formulas for reliability indices of a complex redundant system in the present paper. In 2009, W. L. Wang and G. Q. Xu investigated the well-posedness and stability of the redundant system in [12]. In 2012, F. Zheng and G. T. Zhu showed that the redundant system is exponentially stable by the bounded operators semigroups theory [13]. Based on these results, we discuss the steady-state reliability and show explicit expressions for the steady-state probabilities of the redundant system.

The rest of this paper is organized as follows. In section 2, some results on the well-posedness and stability of the system are reviewed. In section 3, we investigate the steady-state probabilities of the redundant system and obtain the explicit expressions for the stationary availability, mean up-time, mean down-time and mean circle-time. In section 4, several numerical simulation examples are provided to investigate the effects of various system parameters on the reliability indices. Conclusions are drawn in Section 5.

## Mathematical model formulation

The redundant system discussed here comes from [12], which includes more information of the model. Let $a = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3$, $p_0(t)$ be the probability that the system is in operable state at time $t$, $p_1(t)$ the probability that the system is in waiting state at time $t$, $p_i(t, x)dx$ the probability that the system is in operable state at time $t$ and elapsed repair time lies between $x$ and $x+dx$, where $i=2,3,4,5,6$, $p_j(t,x)dx$ the probability that the system is in degraded state at time $t$ and elapsed repair time lies between $x$ and $x+d$x, where $j=8,9,10,11$, and $p_i(t,x)dx$ probability that the system is in failed state at time $t$ and elapsed repair time lies between $x$ and $x+dx$, where $i=7,12$. The governing equations of the system are as follows,

$$\left[\frac{d}{dt}+a\right]p_0(t)=\int_0^\infty p_2(t,x)\mu_1(x)dx+\int_0^\infty p_3(t,x)\mu_2(x)dx+\int_0^\infty p_4(t,x)\mu(x)dx$$
$$+\int_0^\infty \eta(x)[p_5(t,x)+p_6(t,x)]dx+\int_0^\infty p_7(t,x)\mu_3(x)dx+\int_0^\infty p_{12}(t,x)\beta(x)dx, \tag{1}$$

$$\left[\frac{d}{dt}+r_1\right]p_1(t)=\lambda_0\int_0^\infty p_4(t,x)dx, \qquad \left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_3(x)\right]p_7(t,x)=0 \tag{2}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_1(x)+\lambda_2\right]p_2(t,x)=0, \qquad \left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_1(x)+r_2\right]p_8(t,x)=0 \tag{3}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_2(x)+\lambda_1\right]p_3(t,x)=0, \qquad \left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_2(x)+r_2\right]p_9(t,x)=0 \tag{4}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu(x)+\lambda_0\right]p_4(t,x)=0, \qquad \left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_1(x)+\lambda_2\right]p_{10}(t,x)=0 \tag{5}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\eta(x)+\lambda_1+\lambda_2+r_2\right]p_5(t,x)=0, \left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_2(x)+\lambda_1\right]p_{11}(t,x)=0 \tag{6}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\eta(x)+\lambda_1+\lambda_2\right]p_6(t,x)=0, \qquad \left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\beta(x)\right]p_{12}(t,x)=0 \tag{7}$$

and with the boundary conditions

$$p_2(t,0)=\lambda_1p_0(t), p_3(t,0)=\lambda_2p_0(t), p_4(t,0)=\lambda_2\int_0^\infty p_2(t,x)dx+\lambda_1\int_0^\infty p_3(t,x)dx, \tag{8}$$

$$p_5(t,0)=\lambda_0p_0(t)+r_1p_1(t)+\int_0^\infty \mu_1(x)p_8(t,x)dx+\int_0^\infty \mu_2(x)p_9(t,x)dx, \tag{9}$$

$$p_6(t,0)=r_2\int_0^\infty p_5(t,x)dx+\int_0^\infty \mu_1(x)p_{10}(t,x)dx+\int_0^\infty \mu_2(x)p_{11}(t,x)dx, \tag{10}$$

$$p_7(t,0)=\lambda_3p_0(t), \quad p_8(t,0)=\lambda_1\int_0^\infty p_5(t,x)dx, \quad p_9(t,0)=\lambda_2\int_0^\infty p_5(t,x)dx \tag{11}$$

$$p_{10}(t,0)=\lambda_1\int_0^\infty p_6(t,x)dx+r_2\int_0^\infty p_8(t,x)dx, \tag{12}$$

$$p_{11}(t,0)=\lambda_2\int_0^\infty p_6(t,x)dx+r_2\int_0^\infty p_9(t,x)dx, \tag{13}$$

$$p_{12}(t,0)=\lambda_2\int_0^\infty p_{10}(t,x)dx+\lambda_1\int_0^\infty p_{11}(t,x)dx, \tag{14}$$

and the initial conditions $p_0 = 1, p_1 = p_1(0, x) = \cdots = p_{12}(0, x) = 0$.

The equations (1)-(14) can be rewritten as an the abstract Cauchy problem in Banach space $X$

$$(ACP) \quad \begin{cases} \frac{dp(t)}{dt} = \mathcal{A}p(t) & t \geq 0, \\ p(0) = (1, 0, \cdots, 0)^T. \end{cases}$$

in which $\mathcal{A}$ is defined in [12] and $X = \mathbb{R}^2 \times (L^1(\mathbb{R}^+))^{11}$ is Banach space. For $p = (p_0, p_1, p_2(x), \cdots, p_{12}(x)) \in X$, the norm of $p$ is given by $\|p\| = |p_0| + |p_1| + \sum_{i=2}^{12} \int_0^\infty |p_i(x)|dx$.

Let $\mu_4(x) = \mu(x)$, $\mu_5(x) = \beta(x)$ and $\mu_6(x) = \eta(x)$. The main results of the paper [12] are obtained under the following assumptions and they are also valid in this paper.

**General Assumptions**: There exist positive constants $H$ and $c$, such that for any $t \geq 0$, $\int_t^\infty e^{-\int_t^u \mu_i(u)} du > H$ and $c = \min\{\inf_{x \in \mathbb{R}^+} \mu_i(x), i = 1, \cdots, 6\}$.

## Some steady-state reliability indices

According to [5], $p_0(t) + \sum_{i=2}^{6} \int_0^\infty p_i(x,t)dx + \sum_{i=8}^{11} \int_0^\infty p_1(x,t)dx$ and $AV = \lim_{t\to\infty} \big(p_0(t) + \sum_{i=2}^{6} \int_0^\infty p_i(x,t)dx + \sum_{i=8}^{11} \int_0^\infty p_1(x,t)dx\big)$ are the instantaneous availability and the stationary availability of the repairable system, respectively. The availability of the system is one of the most important reliability indices and engineers are especially interested in the steady-state availability. Now we will obtain the stationary availability of the system on the basis of the Theorem 1 and Theorem 2 of [12]. For convenience, set

$$\begin{array}{lll}
f_2(x) = e^{-\int_0^x \mu_1(s)+\lambda_2 ds} & f_3(x) = e^{-\int_0^x \mu_2(s)+\lambda_1 ds} & f_4(x) = e^{-\int_0^x \mu(s)+\lambda_0 ds} \\
f_5(x) = e^{-\int_0^x \eta(s)+\lambda_1+\lambda_2+r_2 ds} & f_6(x) = e^{-\int_0^x \eta(s)+\lambda_1+\lambda_2 ds} & f_7(x) = e^{-\int_0^x \mu_3(s) ds} \\
f_8(x) = e^{-\int_0^x \mu_1(s)+r_2 ds} & f_9(x) = e^{-\int_0^x \mu_2(s)+r_2 ds} & f_{12}(x) = e^{-\int_0^x \beta(s) ds} \\
a_2 = \int_0^\infty \mu_1(x) f_2(x) dx & a_3 = \int_0^\infty \mu_2(x) f_3(x) dx & a_4 = \int_0^\infty \mu(x) f_4(x) dx \\
a_5 = \int_0^\infty \eta(x) f_5(x) dx & a_6 = \int_0^\infty \eta(x) f_6(x) dx & a_8 = \int_0^\infty \mu_1(x) f_8(x) dx \\
a_9 = \int_0^\infty \mu_2(x) f_9(x) dx & b_j = \int_0^\infty f_j(x) dx, & j = 2, 3, \cdots, 9, 12
\end{array}$$

**Theorem 1.** *The stationary availability $AV$ of the system is*

$$AV = \frac{1 + \sum_{i=4}^{6} b_i c_i + \sum_{i=8}^{9} b_i c_i + b_2(c_{10} + \lambda_1) + b_3(c_{11} + \lambda_2)}{1 + c_1 + \sum_{i=4}^{6} b_i c_i + \lambda_3 b_7 + \sum_{i=8}^{9} b_i c_i + b_2(c_{10} + \lambda_1) + b_3(c_{11} + \lambda_2) + b_{12} c_{12}} \tag{15}$$

*in which*

$$\begin{aligned}
&c_1 = r_1^{-1} \lambda_0 \lambda_1 \lambda_2 b_4 (b_2 + b_3), c_4 = \lambda_1 \lambda_2 (b_2 + b_3), \\
&c_5 = [1 - b_5(\lambda_1 a_8 + \lambda_2 a_9)]^{-1} [\lambda_0 + \lambda_0 \lambda_1 \lambda_2 b_4 (b_2 + b_3)], \\
&c_8 = \lambda_1 b_5 [1 - b_5(\lambda_1 a_8 + \lambda_2 a_9)]^{-1} [\lambda_0 + \lambda_0 \lambda_1 \lambda_2 b_4 (b_2 + b_3)], \\
&c_9 = \lambda_2 b_5 [1 - b_5(\lambda_1 a_8 + \lambda_2 a_9)]^{-1} [\lambda_0 + \lambda_0 \lambda_1 \lambda_2 b_4 (b_2 + b_3)], \\
&c_6 = r_2 (b_5 c_5 + a_2 b_8 c_8 + a_3 b_9 c_9) [1 - b_6(\lambda_1 a_2 + \lambda_2 a_3)]^{-1}, \\
&c_{10} = \lambda_1 b_6 c_6 + r_2 b_8 c_8, c_{11} = \lambda_2 b_6 c_6 + r_2 b_9 c_9, c_{12} = \lambda_2 b_2 c_{10} + \lambda_1 b_3 c_{11}.
\end{aligned}$$

Let $p_i$ be the steady-state probability that the system is in state $i$ $(i = 0, \cdots, 12)$, Theorem 1 tells us $p_0 = \|P\|^{-1}$, $p_1 = c_1 \|P\|^{-1}$, $p_2 = \lambda_1 b_2 \|P\|^{-1}$, $p_3 = \lambda_2 b_3 \|P\|^{-1}$, $p_i = c_i b_i \|P\|^{-1}$ $(i = 4, 5, 6)$, $p_7 = \lambda_3 b_7 \|P\|^{-1}$, $p_j = c_j b_j \|P\|^{-1}$ $(j = 8, 9)$, $p_{10} = c_{10} b_2 \|P\|^{-1}$, $p_{11} = c_{11} b_3 \|P\|^{-1}$ and $p_{12} = c_{12} b_{12} \|P\|^{-1}$. With these quantities in hands, we are able to obtain other reliability indices.

**Theorem 2.** *The mean up-time (MUP), mean down-time (MDP) and mean circle-time (MCP) of the redundant system are equal to*

$$MUP = \frac{1 + \sum_{i=4}^{6} b_i c_i + \sum_{i=8}^{9} b_i c_i + b_2(c_{10} + \lambda_1) + b_3(c_{11} + \lambda_2)}{\lambda_0 + \lambda_3 + \lambda_2 b_2 c_{10} + \lambda_1 b_3 c_{11} + r_1 c_4 b_4},$$

$$MDP = \frac{c_1 + b_7 c_7 + b_{12} c_{12}}{\lambda_0 + \lambda_3 + \lambda_2 b_2 c_{10} + \lambda_1 b_3 c_{11} + r_1 c_4 b_4},$$

$$MCP = \frac{\|P\|}{\lambda_0 + \lambda_3 + \lambda_2 b_2 c_{10} + \lambda_1 b_3 c_{11} + r_1 c_4 b_4}.$$

**Proof:** Let $M$ be the steady-state failure frequency of the system, then $M = (\lambda_0 + \lambda_3)p_0 + \lambda_2 p_{10} + \lambda_1 p_{11} + r_1 p_4$ by the definition of the steady-state failure frequency. However, the mean up-time (MUP), mean down-time (MDP) and mean circle-time (MCP) of the redundant system are $\frac{AV}{M}$, $\frac{1-AV}{M}$ and $\frac{1}{M}$, respectively. An easy computation yields these formulations. The proof of the Theorem 2 is complete.

## Numerical Example

In the existing literature (e.g., see [9]-[11]), numerical results are often given under the assumption that the repair time distributions of the units are exponential, that is to say $\mu_i(x)$ are constants. It is not enough to show whether their methods are powerful or not since in this situation it is easy to obtain the reliability indices directly through simple computation. We shall carry out numerical simulations on more broad range. Some numerical results are first illustrated for the above models under the assumption that $\lambda_0 = 0.1, \lambda_1 = 0.2, \lambda_2 = 0.2, \lambda_3 = 0.01, r_1 = 0.05, r_2 = 0.05$; $\mu(x) = \mu = 0.3, \mu_1(x) = \mu_1 = 0.5, \mu_2(x) = \mu_2 = 0.6, \mu_3(x) = \mu_3 = 0.9, \eta(x) = \eta = 0.7$ and

$$\beta(x) = \begin{cases} 0.1, & x \in [0, 10], \\ s, & x \in (10, \infty). \end{cases}$$

The numerical results on the stationary availability, mean up-time, mean down-time and mean circle-time are depicted in Fig.1-Fig.2.

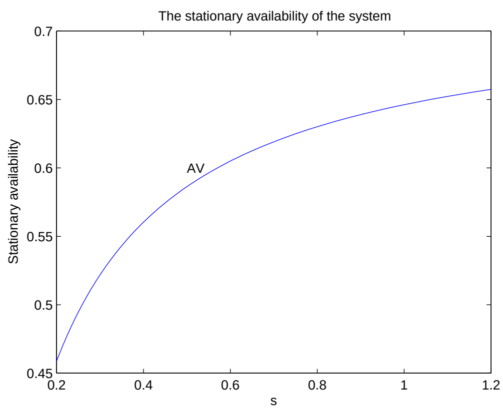


Fig. 1: The stationary availability $AV$ of the system

From the curves of Fig.1-Fig.2 we conclude that the stationary availability increases as $s$ increases. The increase is rapid initially and tends to vanish as $s$ becomes large. The mean circle-time and mean down-time decrease as $s$ increases. The decrease is rapid initially and tends to vanish as $s$ becomes large. However, the mean up-time are invariant as $s$ increases.

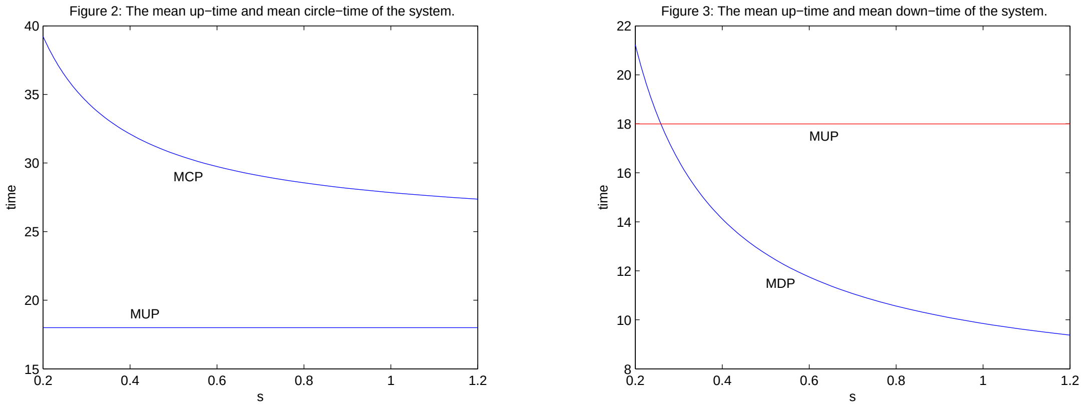


Fig. 2: The mean up-time, mean down-time and mean circle-time of the system

## Conclusion

In this paper, we discussed a complex standby system with a method which is different from the traditional ones and obtained the explicit formulations of the steady-state availability of the system and the system's mean up-time, mean down-time and mean circle-time. Some numerical results were provided to show the effects of various system parameters on the steady-state availability and the mean time.

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