Frequency Weighted H_∞ Model Reduction Based on LMI

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Abstract: This paper treats the problem of a frequency-weighted optimal H_∞ model reduction problem for linear time-invariant (LTI) systems. An algorithm based on the LMI is derived to solve the frequency weighted H_∞ model reduction problem. The aim of the algorithm is to minimize H_∞ norm of the frequency-weighted truncation error between a given LTI system and its lower order approximation. Necessary and sufficient conditions for solving this problem is to meet a series of rank constraints, which generally lead to a non-convex feasibility problem. In addition, it has ensured the stability of reduced-order model when both stable input and output weights are included. Compared with the existing algorithm, the error in this paper is relatively small. An efficient model reduction scheme based on cone complementarity algorithm (CCA) is proposed to solve the non-convex conditions involving rank constraint.

INTRODUCTION

The modeling of a physical system often leads to a high dimensional state-space model, which is hard to implement and analyze. Model reduction is a very important topic in control community. It is important when obtaining a low order model, not to sacrifice vital characteristics of the physical system, such as stability, transient response, steady state error and so on.

Different methods of obtaining reduced order models with these characteristics have been presented over the last thirty years. The main model reduction technique is frequency weighted method which was first introduced by Enns[1,2] and further developed by others[3-6]. This method effectively choose or not mainly depends on the frequency weighting function. Compared with other methods, the linear matrix inequality(LMI) [8,9], as a classical method of data processing has become the main means of parameter estimation and for the reduced order theory. Alternative optimality condition based on the newly developed linear matrix inequality (LMI) machinery was derived in [11] and [10]. Their results originate from the reduced-order H_∞ control perspective and lead to a non-convex feasibility problem in general. Extensions of the optimal unweighted H_∞ model reduction techniques to uncertain linear systems were also reported in.

In this paper, linear matrix inequalities are used for model order reduction and an algorithmic method based on weighted H_∞ norm minimization is described. Different from previous weighted model reduction results, the reduced-order model is shown always stable while stable input and output weighting functions involved.

Design reduced model by LMI

Consider the n^{th} order linear time-invariant continuous systems $T(s)$,and the stable input and output weighting function $W_i(s)$ and $W_o(s)$：

$$T \stackrel{ss}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], W_i \stackrel{ss}{=} \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], W_o \stackrel{ss}{=} \left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right]$$

The purpose is to find a low-order model $T_r(s)$, make

$$\min_{\deg(T_r)\le n_r} \left\| W_o (T(s) - T_r(s)) W_i \right\|_\infty \qquad (1)$$

Here, $A \in R^{n\times n}$, $A_i \in R^{n_i \times n_i}$, $A_o \in R^{n_o \times n_o}$, Assuming that $\begin{bmatrix} B_o \\ D_o \end{bmatrix}$ and $\begin{bmatrix} C_i & D_i \end{bmatrix}$ satisfy the full rank of rows and columns. The following theorem is the main result and proposed a sub-optimal solution to model reduction problem.

Theorem given $g > 0$, As shown in the definition that full order model T, Reduced model T_r and a weighting matrices W_o, W_i. Let $n_w := n + n_i + n_o$ and span the null space of matrices $\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$, $\begin{bmatrix} P_1 & P_2 \end{bmatrix}^T$ respectively (that is $B_o{}^T S_1 + D_o{}^T S_2 = 0$ and $P_1 C_i^T + P_2 D_i^T = 0$). Define

$$A_x := \begin{bmatrix} A & 0 & BC_i P_1^T \\ B_o C & A_o & 0 \\ 0 & 0 & A_i P_1^T + B_i P_2^T \end{bmatrix} \qquad A_y := \begin{bmatrix} A & 0 & BC_i \\ S_1^T B_o C & S_1^T A_o + S_2^T C_o & 0 \\ 0 & 0 & A_i \end{bmatrix}$$

$$B_x = \begin{bmatrix} D_o C & C_o & 0 \end{bmatrix} \qquad B_y = \begin{bmatrix} D_i^T B^T & 0 & B_i^T \end{bmatrix}^T$$

$$X = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & P_1 \end{bmatrix} X \qquad Y = \begin{bmatrix} I & 0 & 0 \\ 0 & S_1^T & 0 \\ 0 & 0 & I \end{bmatrix} Y$$

The presence of the following statements is equivalent：

1、There is a reduced model T_r， $\deg(T_r) \le n_r$，

$$\left\| W_o(s)(T(s) - T_r(s)) W_i(s) \right\|_\infty < g \qquad (2)$$

2、There are matrices $X, Y \in S^{n_w \times n_w}$ satisfy：

$$\begin{bmatrix} A_x^T X^T + X A_x - g\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P_1 P_1^T \end{bmatrix} & B_x \\ B_x^T & -gI \end{bmatrix} < 0 \tag{3}$$

$$\begin{bmatrix} Y^T A_y^T + A_y Y - g\begin{bmatrix} 0 & 0 & 0 \\ 0 & S_1^T S_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & B_y \\ B_y^T & -gI \end{bmatrix} < 0 \tag{4}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \tag{5}$$

$$rank\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n_w + n_r \tag{6}$$

The method of Solving (X,Y) is non-convex optimization when design the reduction model. The problem is available the non-convex optimization described below:

$$\min_{X,Y>0} rank\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \quad \text{subject to} \quad LMI:(3)(4)(5) \tag{7}$$

According to the derivation process of above, the reduced order model can be designed by the following steps:

Step 1: Find X and Y which satisfies the condition (7);

Step 2: Find $X_2 \in R^{n_w \times n_r}$ satisfies $X - Y^{-1} = X_2 X_2{}^T$, where n_r is the order of the reduced order model. Then the matrix W can be obtained by singular value decomposition method of matrix X_2

$$W = \begin{bmatrix} X & X_2{}^T \\ X_2 & I \end{bmatrix} \tag{8}$$

Step 3: Take W into the matrix inequality, we will get a linear matrix inequality which only contains variable K. Then we can get the state space representation of model after order reduction by LMI.

Approximation analysis of a non-convex feasibility problem based on cone complementarity algorithm

According to [7], the principle of cone complementarity algorithm is to associate the rank constraint with the minimum of the trace function $Tr(\mathrm{XY})$. This is the “cone complementarity” problem we proposed. Next step we need to do is to linearize the function $Tr(\mathrm{XY})$ in a local area near the point, since the objective function still is non-convex. Then, we need to find the feasible solution X_0 and Y_0 of the linear matrix inequality.

$$j(X,Y) = Tr(Y_0 Y + X_0 X) \tag{9}$$

Then, the reduced-order model problem is converted to non-convex optimization problem as follows:

$$\min j(X,Y) \quad \text{subject to} \quad LMI:(3)(4)(5) \tag{10}$$

Algorithm 1 For a certain performance index, the proposed algorithm for corresponding reduced-order model is described as follows:

1) Find a feasible point X_0 and Y_0,which satisfy all of the unknown variable of the three matrix inequalities (3), (4), (5) , then set $k=0$;

2) Set $V_k = Y_k, W_k = X_k$, find an optimal solution (X_{k+1}, Y_{k+1}) subject to minimization problem as follow:

$$\min \mathrm{Tr}(V_k X + W_k Y) \quad \text{subject to} \quad LMI:(3)(4)(5) \tag{11}$$

3) Verify the optimal solution whether to meet the matrix inequalities, if yes, exit. Otherwise, check k whether to meet the specified iterations, if it reaches, the system has no solution, otherwise, set $k=k+1$, and go back to step 2);

Algorithm 2 The optimal H_∞ model reduction algorithm

1) For fixed g in $\underline{g} \le g \le \bar{g}$, (11) has no feasible solution when $\underline{g}$ is sufficiently small, or (11) has feasible solution when $\bar{g}$ is sufficiently large;

2) Calculate the mean of $\bar{g}$ and $\underline{g}$, $g_{med} = \dfrac{\bar{g}+\underline{g}}{2}$;

3) Set $g = g_{med}$, solve the equation (15) using algorithm 1. If the feasible solution is none existence, $\underline{g} = g_{med}$, otherwise, $\bar{g} = g_{med}$.

4) For a preestablished accuracy, if $\left|\bar{g} - \underline{g}\right| \le e$, the solution is the optimal reduced order model, the corresponding performance index of optimal H_∞ is $g_{\min} = \bar{g}$, exit. otherwise, go back to 2).

Example

Example: In order to test the model, we consider the fourth order model

$$A=\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & -3 & 0\\ 0 & 0 & 0 & -4\end{bmatrix}\qquad B=\begin{bmatrix}0 & 5\\ 1/2 & -3/2\\ 1 & -5\\ -1/2 & 1/6\end{bmatrix}\qquad C=\begin{bmatrix}1 & 0 & 1 & 0\\ 4/15 & 1 & 0 & 1\end{bmatrix}$$

The input and output weights are given as follow:

$$A_i=A_o=\begin{bmatrix}-4.5 & 0\\ 0 & -4.5\end{bmatrix}\qquad B_i=B_o=\begin{bmatrix}3.0 & 0\\ 0 & 3.0\end{bmatrix}$$

$$C_i=C_o=\begin{bmatrix}1.5 & 0\\ 0 & 1.5\end{bmatrix}\qquad D_i=D_o=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

To the above system, we consider the reduced model by the method in this paper. The results are showed in Fig1, Fig2, Fig3 respectively.

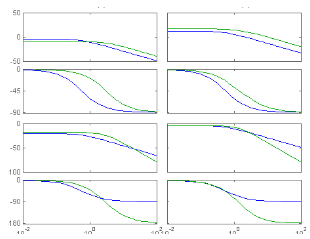

Fig1 Bode diagrams of the first-order model and original model

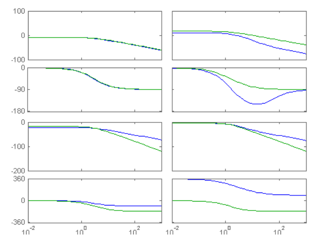

Fig2 Bode diagrams of the second-order model and original model

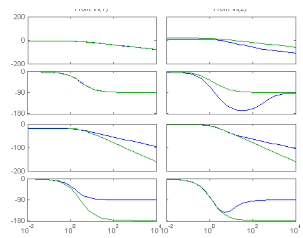

Fig3 Bode diagrams of the third-order model and original model

Fig1, Fig2, Fig3 show the Bode diagrams compared the corresponding first-order, second-order, third-order which by the method we proposed with the original model, where g is 2.2203、0.1030 and 0.1016 respectively. Moreover, the reduced-order error which are calculated respectively by Enns, Lin and Chiu , Wang et al and the method in this paper are tabulated in Table 1.

Table 1 Comparison of the approximation performance between the proposed method and the previous classical methods

The order	Enns	L&C	Wang	Ebw	Error
1	2.1291	2.5744	2.1213	7.2898	2.2203
2	0.2660	0.5607	0.2720	1.4895	0.1030
3	0.1131	0.1645	0.1151	0.3228	0.1016

Conclusion

Aim at the contradiction between the stability and reduced order error in the present of the balanced reduction method of frequency weighted , we propose an improved frequency weighted H_∞ reduction method. Our conclusions can be summarized as follows. The principle of the method is combined the linear matrix inequality with the frequency-weighted, which converted the H_∞ model reduction problem into a convex feasibility problem based on linear matrix inequalities . In this method, it has not only ensured the stability of reduced-order model in Bilateral weighted condition but also has relatively small error compared with the existing algorithm .

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