

Local Progressive Iterative Approximation for Triangular Bézier and Rational Triangular Bézier Surfaces

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Abstract. In this paper, a local PIA format for triangular Bézier surface and rational triangular Bézier surface is designed, which adjusts control points of even permutations only. It generates a surface sequence with finer precision and its limit surface interpolates given data points. Examples show that the surface sequence generated by the local PIA has finer fitting precision.

Introduction

Interpolation and approximation problems of scattered data points are important issues in reverse engineering, and progressive iterative approximation (PIA) is an important method of the interpolation and approximation problems of scattered data points. By adjusting the control points of a surface iteratively, a series of surfaces is generated. If the limit of the series of surfaces interpolates initial control point, we say that the surface hold the PIA property. Researches of totally positive matrix and its properties lay a theoretical foundation for PIA properties. In recent years, there are many studies on the PIA for a curve or a surface. In [2], Hongwei Lin et al. show that the curve (tensor product surface) hold the PIA property so long as the corresponding collocation matrix is nonsingular and the basis is totally positive. And generally speaking, a curve or a surface generated by NTP base holds the PIA property. In [3, 4], the PIA property of the uniform cubic B-spline basis is provided. In [5], a new PIA format for uniform cubic B-spline curve is presented, the PIA format which control points of even indexes interpolate ordered data points, control points of odd indexes interpolate prescribed tangent vectors. And on this basis, Jie Chen, Zhang li et al. [6-8] extend the PIA property of the univariate NTP basis to the bivariate Bernstein basis over a triangle domain. Furthermore, the PIA format has been extended to approximation subdivision surface fitting and many achievements in scientific research have been made. It is showed that Loop subdivision surface, Doo-Sabin subdivision surface, Catmull-Clark subdivision surface hold the PIA property respectively, but the PIA property for Catmull-Clark subdivision surface requires controlling meshes having no vertexes of degree three [9-11].

Most of the interpolation geometric iterative algorithms mentioned above are the so called “global PIA” formats, because they need to adjust all of the control points iteratively. Up to now, there are few studies on the so called “local PIA” formats for surfaces, which need only to adjust a part of control points iteratively. In this paper, a local PIA format for triangular Bézier surface and rational triangular Bézier surface is put forward, which adjusts only control points of even permutations and fixes control points of odd permutations. It generates a surface sequence with finer precision and its limit surface interpolates given data points. Examples show that the surface sequence generated by the local PIA has finer fitting precision.

Preliminaries

Definition1.1 [12] If the sequential order of any two numbers in a permutation is contrary to their size order, namely the foregoing number is larger than the subsequent number, then the two numbers conform a inverse sequence. The total number of all inverse sequences in a permutation is called the inverse number of the permutation. The inverse number of the permutation $j_1 j_2 j_3 \dots j_n$ is

denoted as $t(j_1 j_2 j_3 \mathbf{L} j_n)$.

Definition1.2 [12] The permutation whose inverse number is even is called an even permutation and whose inverse number is odd is called an odd permutation.

Definition1.3 [12] Given two arrays $\mathbf{k} = (k_1, k_2, \mathbf{L}, k_n)$ and $\mathbf{l} = (l_1, l_2, \mathbf{L}, l_n)$, then array \mathbf{k} is arranged before array \mathbf{l} , denote as $\mathbf{k} \mathbf{f} \mathbf{l}$ (or briefly \mathbf{k}, \mathbf{l}), if the first nonzero entry in the difference

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$(k_1 - l_1, k_2 - l_2, \mathbf{L}, k_n - l_n)$ is positive. And this kind order is called lexicographic order.

Local PIA for triangular Bézier surface

Definition2.1 [13] A triangular Bézier surface of degree n over a triangular domain $T := \{(u, v, w) : u, v, w \geq 0, u + v + w = 1\}$ is defined by

$$\mathbf{T}(u, v, w) = \sum_{i+j+k=n} B_{i,j,k}^n(u, v, w) \mathbf{P}_{i,j,k}, \quad (2.1)$$

where $\mathbf{P}_{i,j,k}$ are the control points, $B_{i,j,k}^n(u, v, w) = \frac{n!}{i!j!k!} u^i v^j w^k$ are the bivariate Bernstein basis functions.

In general, there are $(n+2)(n+1)/2$ control points and basis functions in this surface. Therefore, the Bernstein basis functions of degree n over a triangular domain T can be expressed as a $(n+2)(n+1)/2$ -dimensional vector according to the subscripts in lexicographic order:

$$\mathbf{B}^n = (B_{n,0,0}^n, B_{n-1,1,0}^n, B_{n-1,0,1}^n, B_{n-2,2,0}^n, B_{n-2,1,1}^n, B_{n-2,0,2}^n, \mathbf{L}, B_{0,n,0}^n, B_{0,n-1,1}^n, \mathbf{L}, B_{0,1,n-1}^n, B_{0,0,n}^n).$$

Given data points $\mathbf{P}_{i,j,k}$ and triangular Bernstein basis function $B_{i,j,k}^n(u, v, w)$, and denote $\mathbf{P}_{i,j,k}^0 = \mathbf{P}_{i,j,k}^0$, then the initial triangular Bézier surface can be expressed as

$$\mathbf{T}^0(u, v, w) = \sum_{i+j+k=n} B_{i,j,k}^n(u, v, w) \mathbf{P}_{i,j,k}^0.$$

For each control point $\mathbf{P}_{i,j,k}^0$, the first adjusting vector $\mathbf{D}_{i,j,k}^0$ and the control point $\mathbf{P}_{i,j,k}^1$ of the second triangular Bézier surface are computed with following formulae:

$$\mathbf{D}_{i,j,k}^0 = \mathbf{P}_{i,j,k} - \mathbf{T}^0(u, v, w) \text{ and } \mathbf{P}_{i,j,k}^1 = \mathbf{P}_{i,j,k}^0 + \mathbf{D}_{i,j,k}^0,$$

where $i + j + k = n$. Then the second triangular Bézier surface can be expressed as

$$\mathbf{T}^1(u, v, w) = \sum_{i+j+k=n} B_{i,j,k}^n(u, v, w) \mathbf{P}_{i,j,k}^1.$$

Similarly, the $(s+3)$ -triangular Bézier surface can be obtained after the $(s+2)$ -iteration. The adjusting vectors $\mathbf{D}_{i,j,k}^{s+1}$ after the $(s+2)$ -iteration and the control points $\mathbf{P}_{i,j,k}^{s+2}$ of the $(s+3)$ -triangular Bézier surface are computed by formulae:

$$\mathbf{D}_{i,j,k}^{s+1} = \mathbf{P}_{i,j,k} - \mathbf{T}^{s+1}(u, v, w) \text{ and } \mathbf{P}_{i,j,k}^{s+2} = \mathbf{P}_{i,j,k}^{s+1} + \mathbf{D}_{i,j,k}^{s+1}, \quad (2.2)$$

where $i + j + k = n$. So the $(s+3)$ -triangular Bézier surface

$$\mathbf{T}^{s+2}(u, v, w) = \sum_{i+j+k=n} B_{i,j,k}^n(u, v, w) \mathbf{P}_{i,j,k}^{s+2}.$$

According to literature [13], the initial triangular Bézier surface $\mathbf{T}^0(u, v, w)$ has global PIA property. Now we show the local PIA format for triangular Bézier surface based on the lexicographic order.

According to the subscripts in lexicographic order, all the control points in (2.1) can be expressed as : $\mathbf{P}_{n,0,0}, \mathbf{P}_{n-1,1,0}, \mathbf{P}_{n-1,0,1}, \mathbf{P}_{n-2,2,0}, \mathbf{P}_{n-2,1,1}, \mathbf{L}, \mathbf{P}_{0,n,0}, \mathbf{P}_{0,n-1,1}, \mathbf{L}, \mathbf{P}_{0,1,n-1}, \mathbf{P}_{0,0,n}$.

For the set $\{(i, j, k) | i + j + k = n\}$, let the number of its even permutations is M , and the number of its odd permutations is N , then $M + N = (n + 2)(n + 1)/2$.

Denote the sets of the even permutations as $E = \{(i, j, k)_{E_1}, (i, j, k)_{E_2}, \mathbf{L}, (i, j, k)_{E_M}\}$ and the odd permutations as $O = \{(i, j, k)_{O_1}, (i, j, k)_{O_2}, \mathbf{L}, (i, j, k)_{O_N}\}$ respectively. So all of control points of even (odd) permutations in lexicographic order are

$$\mathbf{P}_{(i,j,k)_{E_1}}, \mathbf{P}_{(i,j,k)_{E_2}}, \mathbf{L}, \mathbf{P}_{(i,j,k)_{E_M}} (\mathbf{P}_{(i,j,k)_{O_1}}, \mathbf{P}_{(i,j,k)_{O_2}}, \mathbf{L}, \mathbf{P}_{(i,j,k)_{O_N}}).$$

If we adjust only control points of even permutations and fix control points of odd permutations, then

$$\mathbf{P}_{(i,j,k)_{E_g}}^{s+2} = \mathbf{D}_{(i,j,k)_{E_g}}^{s+1} + \mathbf{P}_{(i,j,k)_{E_g}}^{s+1}, g = 1, 2, \mathbf{L}, M \text{ and } \mathbf{P}_{(i,j,k)_{O_l}}^{s+2} = \mathbf{P}_{(i,j,k)_{O_l}}^{s+1} = \mathbf{P}_{(i,j,k)_{O_l}}, l = 1, 2, \mathbf{L}, N.$$

So for the adjusting vectors of even permutations, according to equation (2.2), we obtain

$$\begin{aligned} \mathbf{D}_{(i,j,k)_{E_g}}^{s+1} &= \mathbf{P}_{(i,j,k)_{E_g}} - \sum_{i+j+k=n} B_{i,j,k}^n (\mathbf{t}_{(i,j,k)_{E_g}}^n) \mathbf{P}_{i,j,k}^{s+1} \\ &= \mathbf{P}_{(i,j,k)_{E_g}} - \sum_{i+j+k=n} B_{i,j,k}^n (\mathbf{t}_{(i,j,k)_{E_g}}^n) \mathbf{P}_{i,j,k}^s - \sum_{(i,j,k)_{E_g} \in E} B_{i,j,k}^n (\mathbf{t}_{(i,j,k)_{E_g}}^n) \mathbf{D}_{i,j,k}^s \\ &= \mathbf{D}_{(i,j,k)_{E_g}}^s - \mathbf{D}_{(i,j,k)_{E_1}}^s B_{(i,j,k)_{E_1}}^n (\mathbf{t}_{(i,j,k)_{E_g}}^n) - \mathbf{D}_{(i,j,k)_{E_2}}^s B_{(i,j,k)_{E_2}}^n (\mathbf{t}_{(i,j,k)_{E_g}}^n) - \mathbf{L} \\ &\quad - \mathbf{D}_{(i,j,k)_{E_M}}^s B_{(i,j,k)_{E_M}}^n (\mathbf{t}_{(i,j,k)_{E_g}}^n) (g = 1, 2, \mathbf{L}, M). \end{aligned} \quad (2.3)$$

Similarly, for the adjusting vectors of odd permutations, we have

$$\begin{aligned} \mathbf{D}_{(i,j,k)_{O_l}}^{s+1} &= \mathbf{P}_{(i,j,k)_{O_l}} - \sum_{i+j+k=n} B_{i,j,k}^n (\mathbf{t}_{(i,j,k)_{O_l}}^n) \mathbf{P}_{i,j,k}^{s+1} \\ &= \mathbf{D}_{(i,j,k)_{O_l}}^s - \mathbf{D}_{(i,j,k)_{E_1}}^s B_{(i,j,k)_{E_1}}^n (\mathbf{t}_{(i,j,k)_{O_l}}^n) - \mathbf{D}_{(i,j,k)_{E_2}}^s B_{(i,j,k)_{E_2}}^n (\mathbf{t}_{(i,j,k)_{O_l}}^n) - \mathbf{L} \\ &\quad - \mathbf{D}_{(i,j,k)_{E_M}}^s B_{(i,j,k)_{E_M}}^n (\mathbf{t}_{(i,j,k)_{O_l}}^n) (l = 1, 2, \mathbf{L}, N). \end{aligned} \quad (2.4)$$

Denote $\mathbf{D}^{s+1} = (\mathbf{D}_{(i,j,k)_{E_1}}^{s+1}, \mathbf{D}_{(i,j,k)_{E_2}}^{s+1}, \mathbf{L}, \mathbf{D}_{(i,j,k)_{E_M}}^{s+1}, \mathbf{D}_{(i,j,k)_{O_1}}^{s+1}, \mathbf{D}_{(i,j,k)_{O_2}}^{s+1}, \mathbf{L}, \mathbf{D}_{(i,j,k)_{O_N}}^{s+1})^T$, then from equation (2.2), we obtain

$$\mathbf{D}^{s+1} = \mathbf{B} \mathbf{D}^s, \mathbf{B} = \begin{bmatrix} I_{M \times M} - \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{B}_1 &= \begin{bmatrix} B_{(i,j,k)_{E_1}}^n (\mathbf{t}_{(i,j,k)_{E_1}}^n) & B_{(i,j,k)_{E_2}}^n (\mathbf{t}_{(i,j,k)_{E_1}}^n) & \mathbf{L} & B_{(i,j,k)_{E_M}}^n (\mathbf{t}_{(i,j,k)_{E_1}}^n) \\ B_{(i,j,k)_{E_1}}^n (\mathbf{t}_{(i,j,k)_{E_2}}^n) & B_{(i,j,k)_{E_2}}^n (\mathbf{t}_{(i,j,k)_{E_2}}^n) & \mathbf{L} & B_{(i,j,k)_{E_M}}^n (\mathbf{t}_{(i,j,k)_{E_2}}^n) \\ \mathbf{M} & \mathbf{M} & & \mathbf{M} \\ B_{(i,j,k)_{E_1}}^n (\mathbf{t}_{(i,j,k)_{E_M}}^n) & B_{(i,j,k)_{E_2}}^n (\mathbf{t}_{(i,j,k)_{E_M}}^n) & \mathbf{L} & B_{(i,j,k)_{E_M}}^n (\mathbf{t}_{(i,j,k)_{E_M}}^n) \end{bmatrix}, \mathbf{B}_2 = \mathbf{O}_{M \times N}, \\ \mathbf{B}_3 &= \begin{bmatrix} -B_{(i,j,k)_{E_1}}^n (\mathbf{t}_{(i,j,k)_{O_1}}^n) & -B_{(i,j,k)_{E_2}}^n (\mathbf{t}_{(i,j,k)_{O_1}}^n) & \mathbf{L} & -B_{(i,j,k)_{E_M}}^n (\mathbf{t}_{(i,j,k)_{O_1}}^n) \\ -B_{(i,j,k)_{E_1}}^n (\mathbf{t}_{(i,j,k)_{O_2}}^n) & -B_{(i,j,k)_{E_2}}^n (\mathbf{t}_{(i,j,k)_{O_2}}^n) & \mathbf{L} & -B_{(i,j,k)_{E_M}}^n (\mathbf{t}_{(i,j,k)_{O_2}}^n) \\ \mathbf{M} & \mathbf{M} & & \mathbf{M} \\ -B_{(i,j,k)_{E_1}}^n (\mathbf{t}_{(i,j,k)_{O_N}}^n) & -B_{(i,j,k)_{E_2}}^n (\mathbf{t}_{(i,j,k)_{O_N}}^n) & \mathbf{L} & -B_{(i,j,k)_{E_M}}^n (\mathbf{t}_{(i,j,k)_{O_N}}^n) \end{bmatrix}, \mathbf{B}_4 = \mathbf{I}_{N \times N}. \end{aligned}$$

Because \mathbf{B} is a reducible matrix, so the adjusting control points and fixing control points can be disposed individually.

Theorem1. If the matrix B_1 is nonsingular, then $\lim_{s \rightarrow \infty} \mathbf{P}_{(i,j,k)_{E_g}}^{s+1} = \mathbf{P}_{(i,j,k)_{E_g}}, g = 1, 2, \mathbf{L}, M$.

Proof. From equation (2.3), the adjusting vectors of even permutations can be expressed as:

$$\mathbf{D}_{(i,j,k)_{E_g}}^{s+1} = \mathbf{D}_{(i,j,k)_{E_g}}^s - \sum_{(i,j,k)_{E_g} \in E} \mathbf{D}_{i,j,k}^s \mathbf{B}_{i,j,k}^n (\mathbf{t}_{(i,j,k)_{E_g}}^n), g = 1, 2, \mathbf{L}, M.$$

Denote $\mathbf{D}_E^{s+1} = (\mathbf{D}_{(i,j,k)_{E_1}}^{s+1}, \mathbf{D}_{(i,j,k)_{E_2}}^{s+1}, \mathbf{L}, \mathbf{D}_{(i,j,k)_{E_M}}^{s+1})^T$, then we have

$$\mathbf{D}_E^{s+1} = (I - B_1) \mathbf{D}_E^s = (I - B_1)^{s+1} \mathbf{D}_E^0.$$

According to literature [13], the spectral radius of matrix B_1 satisfies $0 < p(B_1) \leq 1$, that is to say $0 < p(I - B_1) \leq 1$. so when $s \rightarrow \infty$, we have $\mathbf{D}_{(i,j,k)_{E_g}}^{s+1} \rightarrow 0$.

Then from equation (2.3), we obtain $\lim_{s \rightarrow \infty} \mathbf{P}_{(i,j,k)_{E_g}}^{s+1} = \mathbf{P}_{(i,j,k)_{E_g}}, g = 1, 2, \mathbf{L}, M$.

The nonsingular matrix B_1 is a principal sub-matrix of the totally positive collocation matrix of the normalized totally positive basis $B_{i,j,k}^n(u, v, w) = \frac{n!}{i!j!k!} u^i v^j w^k$, so B_1 is a nonsingular totally positive matrix, and all of its eigenvalues satisfy $0 < I(B_1) \leq 1$.

Denote the eigenvalues of the matrix $I - B_1$ as $I_1, I_2, I_3, \mathbf{L}, I_M$, since B_1 is a totally positive matrix, there exists the invertible matrix Y , such that

$$I - B_1 = Y^{-1} \text{diag}(I_1, I_2, I_3, \mathbf{L}, I_M) Y.$$

Denote $\mathbf{D}_O^{s+1} = (\mathbf{D}_{(i,j,k)_{O_1}}^{s+1}, \mathbf{D}_{(i,j,k)_{O_2}}^{s+1}, \mathbf{L}, \mathbf{D}_{(i,j,k)_{O_N}}^{s+1})^T$, $\mathbf{D}_E^{s+1} = (\mathbf{D}_{(i,j,k)_{E_1}}^{s+1}, \mathbf{D}_{(i,j,k)_{E_2}}^{s+1}, \mathbf{L}, \mathbf{D}_{(i,j,k)_{E_M}}^{s+1})^T$. According to equation (2.2), equation (2.4) can be expanded:

$$\begin{aligned} \mathbf{D}_{(i,j,k)_{O_l}}^{s+1} &= \mathbf{D}_{(i,j,k)_{O_l}}^{s-1} - \mathbf{D}_{(i,j,k)_{E_1}}^{s-1} \mathbf{B}_{(i,j,k)_{E_1}}^n (\mathbf{t}_{(i,j,k)_{O_l}}^n) - \mathbf{L} - \mathbf{D}_{(i,j,k)_{E_M}}^{s-1} \mathbf{B}_{(i,j,k)_{E_M}}^n (\mathbf{t}_{(i,j,k)_{O_l}}^n) \\ &\quad - \mathbf{D}_{(i,j,k)_{E_1}}^s \mathbf{B}_{(i,j,k)_{E_1}}^n (\mathbf{t}_{(i,j,k)_{O_l}}^n) - \mathbf{L} - \mathbf{D}_{(i,j,k)_{E_M}}^s \mathbf{B}_{(i,j,k)_{E_M}}^n (\mathbf{t}_{(i,j,k)_{O_l}}^n) \\ &= \mathbf{D}_{(i,j,k)_{O_l}}^0 - \mathbf{B}_{(i,j,k)_{E_1}}^n (\mathbf{t}_{(i,j,k)_{O_l}}^n) \sum_{h=0}^s \mathbf{D}_{(i,j,k)_{E_1}}^h - \mathbf{L} - \mathbf{B}_{(i,j,k)_{E_M}}^n (\mathbf{t}_{(i,j,k)_{O_l}}^n) \sum_{h=0}^s \mathbf{D}_{(i,j,k)_{E_M}}^h, \end{aligned}$$

So we have

$$\mathbf{D}_O^{s+1} = \mathbf{D}_O^0 + B_3 \sum_{h=0}^s \mathbf{D}_E^h. \quad (2.5)$$

Substituting $\mathbf{D}_E^h = (I - B_1) \mathbf{D}_E^{h-1} = Y^{-1} \text{diag}(I_1^h, I_2^h, \mathbf{L}, I_M^h) Y \mathbf{D}_E^0$ into equation (2.5), yields

$$\mathbf{D}_O^{s+1} = \mathbf{D}_O^0 + B_3 Y^{-1} \text{diag}\left(\frac{1 - I_1^{s+1}}{1 - I_1}, \frac{1 - I_2^{s+1}}{1 - I_2}, \mathbf{L}, \frac{1 - I_M^{s+1}}{1 - I_M}\right) Y \mathbf{D}_E^0, \quad (2.6)$$

where $I_1, I_2, \mathbf{L}, I_M \in (0, 1]$.

Theorem 2. When $s \rightarrow \infty$, the limit of the adjusting vectors \mathbf{D}_O^{s+1} of odd permutations is

$$\mathbf{D}_O = \mathbf{D}_O^0 + B_3 Y^{-1} \text{diag}\left(\frac{1}{1 - I_1}, \frac{1}{1 - I_2}, \mathbf{L}, \frac{1}{1 - I_M}\right) Y \mathbf{D}_E^0.$$

Proof. This result is obtained by taking limits from both sides of equation (2.6).

Local PIA for rational triangular Bézier surface

Definition 3.1[13] A rational triangular Bézier surface of degree m over a triangle domain $T := \{(u, v, w) : u, v, w \geq 0, u + v + w = 1\}$ is defined by

$$R(u, v, w) = \frac{\sum_{i+j+k=m} w_{i,j,k} B_{i,j,k}^m(u, v, w) Q_{i,j,k}}{\sum_{i+j+k=m} w_{i,j,k} B_{i,j,k}^m(u, v, w)}, \quad (3.1)$$

where $w_{i,j,k}$ are the weights, $Q_{i,j,k}$ are the control points, $B_{i,j,k}^m(u, v, w) = \frac{m!}{i!j!k!} u^i v^j w^k$ are the bivariate Bernstein basis functions.

Equation (3.1) can be transformed into

$$R(u, v, w) = \sum_{i+j+k=m} r_{i,j,k}^m(u, v, w) Q_{i,j,k},$$

$$\text{where } r_{i,j,k}^m(u, v, w) = \frac{w_{i,j,k} B_{i,j,k}^m(u, v, w)}{\sum_{i+j+k=m} w_{i,j,k} B_{i,j,k}^m(u, v, w)}.$$

Similar to section 2, according to the subscripts in lexicographic order, all the control points can be expressed as: $Q_{m,0,0}, Q_{m-1,1,0}, Q_{m-1,0,1}, Q_{m-2,2,0}, \mathbf{L}, Q_{0,m,0}, Q_{0,m-1,1}, \mathbf{L}, Q_{0,1,m-1}, Q_{0,0,m}$.

For the set $\{(i, j, k) \mid i + j + k = m\}$, let the number of its even permutations is F , and the number of its odd permutations is G , then $F + G = (m + 2)(m + 1)/2$.

Denote the sets of the even permutations as $E' = \{(i, j, k)_{E'_1}, (i, j, k)_{E'_2}, \mathbf{L}, (i, j, k)_{E'_F}\}$ and the odd permutations as $O' = \{(i, j, k)_{O'_1}, (i, j, k)_{O'_2}, \mathbf{L}, (i, j, k)_{O'_G}\}$ respectively. So all of control points of even (odd) permutations in lexicographic order are

$$Q_{(i,j,k)_{E'_1}}, Q_{(i,j,k)_{E'_2}}, \mathbf{L}, Q_{(i,j,k)_{E'_F}}, Q_{(i,j,k)_{O'_1}}, Q_{(i,j,k)_{O'_2}}, \mathbf{L}, Q_{(i,j,k)_{O'_G}}.$$

If we adjust only control points of even permutations, fix control points of odd permutations, then

$$Q_{(i,j,k)_{O'_d}}^{s+1} = Q_{(i,j,k)_{O'_d}}^{s+1} = Q_{(i,j,k)_{O'_d}}, d = 1, 2, \mathbf{L}, G \text{ and } Q_{(i,j,k)_{E'_c}}^{s+1} = D_{(i,j,k)_{E'_c}}^{s+1} + Q_{(i,j,k)_{E'_c}}^{s+1}, c = 1, 2, \mathbf{L}, F.$$

So for the adjusting vectors of even permutations, we obtain

$$\begin{aligned} D_{(i,j,k)_{E'_c}}^{s+1} &= Q_{(i,j,k)_{E'_c}}^{s+1} - \sum_{i+j+k=m} r_{i,j,k}^m(t_{(i,j,k)_{E'_c}}^m) Q_{i,j,k}^{s+1} \\ &= D_{(i,j,k)_{E'_c}}^s - D_{(i,j,k)_{E'_1}}^s r_{(i,j,k)_{E'_1}}^m(t_{(i,j,k)_{E'_1}}^m) - D_{(i,j,k)_{E'_2}}^s r_{(i,j,k)_{E'_2}}^m(t_{(i,j,k)_{E'_2}}^m) \\ &\quad \mathbf{L} - D_{(i,j,k)_{E'_F}}^s r_{(i,j,k)_{E'_F}}^m(t_{(i,j,k)_{E'_F}}^m), \end{aligned} \quad (3.2)$$

where $c = 1, 2, \mathbf{L}, F$. And for the adjusting vectors of odd permutations, we have

$$\begin{aligned} D_{(i,j,k)_{O'_d}}^{s+1} &= Q_{(i,j,k)_{O'_d}}^{s+1} - \sum_{i+j+k=m} r_{i,j,k}^m(t_{(i,j,k)_{O'_d}}^m) Q_{i,j,k}^{s+1} \\ &= D_{(i,j,k)_{O'_d}}^s - D_{(i,j,k)_{O'_1}}^s r_{(i,j,k)_{O'_1}}^m(t_{(i,j,k)_{O'_1}}^m) - D_{(i,j,k)_{O'_2}}^s r_{(i,j,k)_{O'_2}}^m(t_{(i,j,k)_{O'_2}}^m) - \mathbf{L} \\ &\quad - D_{(i,j,k)_{O'_F}}^s r_{(i,j,k)_{O'_F}}^m(t_{(i,j,k)_{O'_F}}^m) \end{aligned} \quad (3.3)$$

where $d = 1, 2, \mathbf{L}, G$

Denote $D^{s+1} = (D_{(i,j,k)_{E'_1}}^{s+1}, D_{(i,j,k)_{E'_2}}^{s+1}, \mathbf{L}, D_{(i,j,k)_{E'_F}}^{s+1}, D_{(i,j,k)_{O'_1}}^{s+1}, D_{(i,j,k)_{O'_2}}^{s+1}, \mathbf{L}, D_{(i,j,k)_{O'_G}}^{s+1})^T$, then from equations (3.2) and (3.3), we obtain

$$D^{s+1} = D D^s, D = \begin{bmatrix} I_{F \times F} - D_1 & D_2 \\ D_3 & D_4 \end{bmatrix},$$

where

$$D_1 = \begin{bmatrix} r_{(i,j,k)_{E_1}}^m(\mathbf{t}_{(i,j,k)_{E_1}}^m) & r_{(i,j,k)_{E_2}}^m(\mathbf{t}_{(i,j,k)_{E_1}}^m) & \mathbf{L} & r_{(i,j,k)_{E_F}}^m(\mathbf{t}_{(i,j,k)_{E_1}}^m) \\ r_{(i,j,k)_{E_1}}^m(\mathbf{t}_{(i,j,k)_{E_2}}^m) & r_{(i,j,k)_{E_2}}^m(\mathbf{t}_{(i,j,k)_{E_2}}^m) & \mathbf{L} & r_{(i,j,k)_{E_F}}^m(\mathbf{t}_{(i,j,k)_{E_2}}^m) \\ \mathbf{M} & \mathbf{M} & & \mathbf{M} \\ r_{(i,j,k)_{E_1}}^m(\mathbf{t}_{(i,j,k)_{E_F}}^m) & r_{(i,j,k)_{E_2}}^m(\mathbf{t}_{(i,j,k)_{E_F}}^m) & \mathbf{L} & r_{(i,j,k)_{E_F}}^m(\mathbf{t}_{(i,j,k)_{E_F}}^m) \end{bmatrix}, D_2 = O_{F \times G},$$

$$D_3 = \begin{bmatrix} -r_{(i,j,k)_{E_1}}^m(\mathbf{t}_{(i,j,k)_{O_1}}^m) & -r_{(i,j,k)_{E_2}}^m(\mathbf{t}_{(i,j,k)_{O_1}}^m) & \mathbf{L} & -r_{(i,j,k)_{E_F}}^m(\mathbf{t}_{(i,j,k)_{O_1}}^m) \\ -r_{(i,j,k)_{E_1}}^m(\mathbf{t}_{(i,j,k)_{O_2}}^m) & -r_{(i,j,k)_{E_2}}^m(\mathbf{t}_{(i,j,k)_{O_2}}^m) & \mathbf{L} & -r_{(i,j,k)_{E_F}}^m(\mathbf{t}_{(i,j,k)_{O_2}}^m) \\ \mathbf{M} & \mathbf{M} & & \mathbf{M} \\ -r_{(i,j,k)_{E_1}}^m(\mathbf{t}_{(i,j,k)_{O_G}}^m) & -r_{(i,j,k)_{E_2}}^m(\mathbf{t}_{(i,j,k)_{O_G}}^m) & \mathbf{L} & -r_{(i,j,k)_{E_F}}^m(\mathbf{t}_{(i,j,k)_{O_G}}^m) \end{bmatrix}, D_4 = I_{G \times G}.$$

Because D is a reducible matrix, so the adjusting control points and fixing control points can be disposed individually.

Similar to the proof of theorem 1, theorem 3 below can be obtained.

Theorem3. If the matrix D_1 is nonsingular, then $\lim_{s \rightarrow \infty} \mathbf{Q}_{(i,j,k)_{E_c}}^{s+1} = \mathbf{Q}_{(i,j,k)_{E_c}}, c = 1, 2, \mathbf{L}, F$.

Denote $\mathbf{D}_{E'}^{s+1} = (\mathbf{D}_{(i,j,k)_{E_1}}^{s+1}, \mathbf{D}_{(i,j,k)_{E_2}}^{s+1}, \mathbf{L}, \mathbf{D}_{(i,j,k)_{E_F}}^{s+1})^T, \mathbf{D}_{O'}^{s+1} = (\mathbf{D}_{(i,j,k)_{O_1}}^{s+1}, \mathbf{D}_{(i,j,k)_{O_2}}^{s+1}, \mathbf{L}, \mathbf{D}_{(i,j,k)_{O_G}}^{s+1})^T$. The eigenvalues of the matrix $I - D_1$ as $I'_1, I'_2, \mathbf{L}, I'_F$, exists the invertible X , such that

$$I - D_1 = X^{-1} \text{diag}(I'_1, I'_2, \mathbf{L}, I'_F) X.$$

Similar to the proof of theorem 2, theorem 4 below can be obtained

Theorem 4. When $s \rightarrow \infty$, the limit of the adjusting vectors $\mathbf{D}_{O'}^{s+1}$ of odd permutations is

$$\mathbf{D}_{O'} = \mathbf{D}_{O'}^0 + D_3 X^{-1} \text{diag}(\frac{1}{1-I'_1}, \frac{1}{1-I'_2}, \mathbf{L}, \frac{1}{1-I'_F}) X \mathbf{D}_{E'}^0.$$

Numerical examples and error analysis

Example4.1 Given a triangular Bézier surface of degree 2, its control points in lexicographic order are (0.1, 3.1, 0.5), (−0.4, 2.3, 3), (0.5, 2, 4), (−1.1, 1.5, 1), (0.2, 1.4, 3), (1.2, 1.5, 0.5).

The initial triangular Bézier surface of degree 2, the surface which adjusts all of control points after the fifth iteration (global PIA) and the surface which adjusts control points of even permutations and fixes control points of odd permutations after the fifth iteration (local PIA) are shown in Figs.1-3, respectively. Under the same of iteration times, error analysis and comparisons are shown in Table 1.

Example4.2 Given a rational triangular Bézier surface of degree 2, its control points in lexicographic order are (−5, 0, 0), (0, −4, 0), (0, 0, −3), (0, 0, 3), (0, 4, 0), (5, 0, 0).

The initial rational triangular Bézier surface of degree 2, the surface which adjusts all of control points after the fifth iteration (global PIA) and the surface which adjusts control points of even permutations and fixes control points of odd permutations after the fifth iteration (local PIA) with the weight $w = 3$ are shown in Fig.4-6, respectively. Under the same of iteration times, error analysis and comparisons are shown in Table 2.

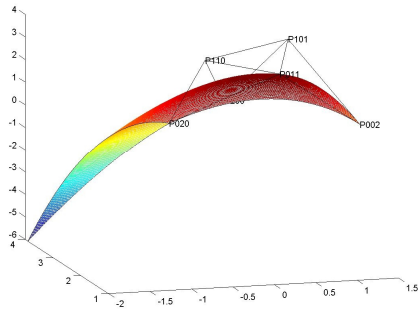


Fig.1. Initial triangular Bézier surface.

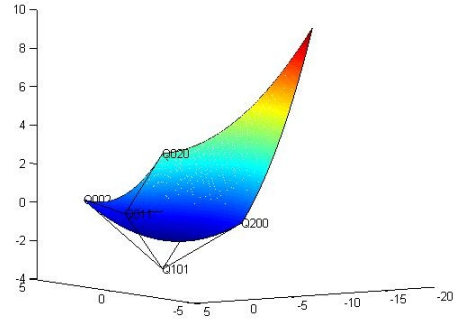


Fig.4. Initial rational triangular Bézier surface.

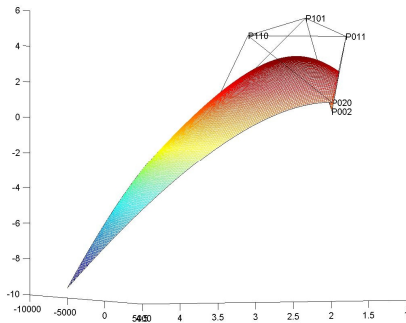


Fig.2. The surface which adjusts all of control points after the fifth iteration.

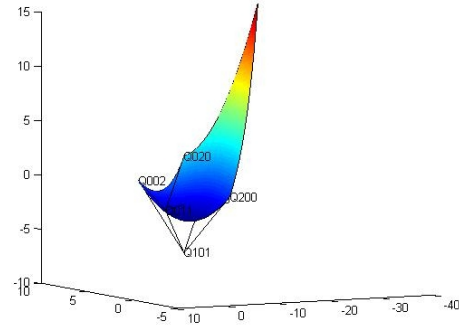


Fig.5. The surface which adjusts all of control points after the fifth iteration.

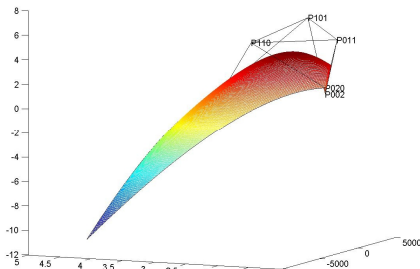


Fig.3. The surface which adjusts only control points of even permutations after the fifth iteration.

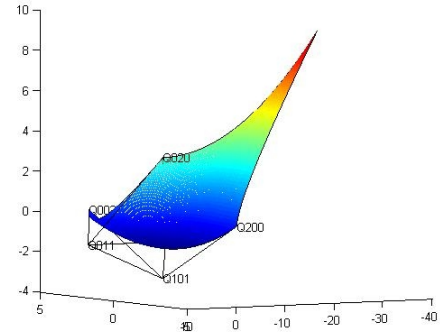


Fig.6. The surface which adjusts only control points of even permutations after the fifth iteration.

Table 1

PIA	Iteration times		
	5	10	15
global PIA	3.5302e-04	3.3615e-07	3.2802e-10
Local PIA	1.8094e-04	1.7446e-07	1.7030e-10

Table 2

PIA	Iteration times		
	5	10	15
Global PIA	9.3572e-04	8.8776e-07	8.6616e-10
local PIA	5.8096e-04	5.5357e-07	5.4018e-10

Summary

In this paper, a local PIA for triangular Bézier and rational triangular Bézier surfaces over a triangle domain is designed, which adjusts control points of even permutations only. It is showed that the local PIA is convergent. Examples of quadratic triangular Bézier and quadratic rational triangular Bézier surfaces show that the local PIA generates surface with finer precision, it converges more quickly than global PIA. Similarly, we can discuss the local PIA for triangular Bézier and rational triangular Bézier surfaces over a triangle domain which adjusts control points of

odd permutations only, and more generally adjusts only any subset of all control points.

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