Steady-State Kalman Estimator for Descriptor Systems with Colored Noise

Yan Xu and Guosheng Zhang

Beijing Institute of Graphic Communication, Beijing 102600, China

Abstract—**Using the modern time series analysis method in the time domain, based on the ARMAinnovation model, a steady-state Kalman estimator for descriptor systems with colored noise is introduced,and employing the state observer principle, the pole-assignment descriptor steady-state Kalman estimator is also presented. They have global asymptotic stability and can handle the filtering, smoothing and prediction problems in unified frameworks, thus avoiding the solution of the Riccatiequations.**

Keywords-escriptor systems; colored noise; steady-state Kalman estimator; global asymptoticstability

I. INTRODUCTION

Descriptor systems often occur in many fields including circuit, economics, robotics, etc, and have attracted considerable attention in recent years[1]. Steady-stateKalman estimators fordescriptor systems in [2] and [3]are limited to white noises, and the noises of the practical problems are often not the ideal white noises. The descriptor steady-state Kalman estimator of [2]have a shortcoming: it may be non-asymptotically stableand the optimal initial valueis required. In this paper, a steady-state Kalman estimator for descriptor systems with colored noise is given. It overcomes the limitations of white noise. In order to obtain the global asymptotic stability of the estimator, the pole-assignment descriptor steady-state Kalman estimator is presented by using the principle of the state observer. Not only it has the global asymptotic stability, but also the effect of theinitial state estimate can be rapidly forgotten by assigning the pole of the estimator.

Consider the discrete time descriptor stochastic system

$$Mx(t+1) = \Phi x(t) + \Gamma w(t) \tag{1}$$

$$y(t) = Hx(t) + v(t) \tag{2}$$

where the state $x(t) \in R^n$, the measurement $y(t) \in R^m$, $w(t) \in R^r$, $v(t) \in R^m$, M, Φ, Γ and H are the constant matrices.

Assumption 1

The $n \times n$ matrix M is singular, that is

$$\det(M) = 0.$$

Assumption 2

The system is completely observable, that is

$$\operatorname{rank}\begin{bmatrix} zM - \Phi \\ H \end{bmatrix} = n,$$

$$\forall z \in C; \quad \operatorname{rank}\begin{bmatrix} M \\ H \end{bmatrix} = n. \tag{3}$$

Assumption 3

The system is regular, that is

$$\forall z \in C, \ \det(zM - \Phi) \not\equiv 0.$$

Assumption 4

$v(t)$ is the white noise with zero mean, and $w(t)$ is the zero meancolored noise correlated with $v(t)$, and

$$\mathrm{E}\big[w(t)w^\mathrm{T}(t)\big] = W_0,$$

$$\mathrm{E}\big[w(t)w^\mathrm{T}(t+1)\big] = W_1,$$

$$\mathrm{E}\big[w(t)w^\mathrm{T}(t+j)\big] = 0 \ (j \geq 2),$$

$$\mathrm{E}\big[w(t)v^\mathrm{T}(t)\big] = S_0,$$

$$\mathrm{E}\big[w(t)v^\mathrm{T}(t+1)\big] = S_1,$$

$$\mathrm{E}\big[v(t)v^\mathrm{T}(j)\big] = V\delta_{tj} \ (\delta_{tt} = 1, \delta_{tj} = 0 (t \neq j)),$$

$$\mathrm{E}\big[w(t)v^\mathrm{T}(t+j)\big] = 0 \ (j \geq 2),$$

$$\mathrm{E}\big[w(t)v^\mathrm{T}(t-j)\big] = 0 \ (j \geq 1). \tag{4}$$

II. ARMA INNOVATION MODEL

From (1)and (2)we have

$$y(t) = H(M - q^{-1}\Phi)^{-1}\Gamma q^{-1}w(t) + v(t), \tag{5}$$

where q^{-1} is the backward shift operator. Introducing the leftcoprimefactorization

$$H(M - q^{-1}\Phi)^{-1}\Gamma q^{-1} = A^{-1}Bq^\tau, \tag{6}$$

where A and B are polynomial matrices having the form

$$X = X(q^{-1}) = X_0 + X_1 q^{-1} + \Lambda + X_{n_x} q^{-n_x},$$

$$A_0 = I_m, B_0 \neq 0,$$

τ is an integer.

Substituting (6)into (5)yields the ARMA innovation model

$$Ay(t) = De(t), \tag{7}$$

$$De(t) = Bq^\tau w(t) + Av(t), \tag{8}$$

where D is a stable polynomial matrix, $D_0 = I_m$. $e(t) \in R^m$ is thewhite noise with zero mean and variance matrix Q_e. D and Q_e can be obtained by using the Gevers-Wouters algorithm[4]. According to (7), the innovations $e(t)$ can be computed recursively as

$$e(t) = Ay(t) - D_1 e(t-1) - \Lambda - D_{n_d} e(t - n_d),$$

$$t = n_d, n_d + 1\Lambda. \tag{9}$$

with the initial values $(e(0), \Lambda, e(n_d - 1))$ .And the spectral matrix of the random process in (8)is assumed nonsingular[5].

III. Noise Estimators

Theorem 1

Under Assumptions 1~4, the system (1)~(2) have

$$E\left[w(t)e^{\mathrm{T}}(j)\right] = J_{j-t},$$

$$E\left[v(t)e^{\mathrm{T}}(j)\right] = L_{j-t}, \tag{10}$$

then we have the colored noiseestimators

$$\hat{w}(t \mid t + N) = \sum_{i=-(\tau \vee 0)-1}^{N} J_i Q_e^{-1} e(t+i),$$

$$\hat{v}(t \mid t + N) = \sum_{i=-(\tau \vee 0)-1}^{N} L_i Q_e^{-1} e(t+i). \tag{11}$$

where

$$J_i = W_0 F_{i+(\tau \vee 0)}^{\mathrm{T}} + W_1 F_{i+(\tau \vee 0)-1}^{\mathrm{T}} + W_1^{\mathrm{T}} F_{i+(\tau \vee 0)+1}^{\mathrm{T}}$$
$$+ S_0 G_{i+(\tau \vee 0)}^{\mathrm{T}} + S_1 F_{i+(\tau \vee 0)-1}^{\mathrm{T}} \ (t \geq 1),$$

$$L_i = S_0^{\mathrm{T}} F_{i+(\tau \vee 0)}^{\mathrm{T}} + S_1^{\mathrm{T}} F_{i+(\tau \vee 0)+1}^{\mathrm{T}} + V G_{i+(\tau \vee 0)}^{\mathrm{T}}, \tag{12}$$

where

$$(a \vee b) = \max(a, b),$$

$$(a \wedge b) = \min(a, b),$$

and F_i, G_i can be computed recursively as

$$F_i = -D_1 F_{i-1} - \Lambda \Lambda - D_{n_d} F_{i-n_d} + \overline{B}_i,$$

$$F_i = 0 \ (i < 0), \ \overline{B}_i = 0 \ (i > n_{\overline{b}})$$

$$G_i = -D_1 G_{i-1} - \Lambda \Lambda - D_{n_d} G_{i-n_d} + \overline{A}_i,$$

$$G_i = 0 \ (i < 0), \ \overline{A}_i = 0 \ (i > n_{\overline{a}}), \tag{13}$$

And

$$\overline{B}(q^{-1}) = B(q^{-1}) q^{(\tau \wedge 0)},$$

$$\overline{A}(q^{-1}) = A(q^{-1}) q^{(-\tau \wedge 0)}.$$

Proof.

According to the proof method of [4] under Assumption 4, we get

$$E\left[w(t)e^{\mathrm{T}}(j)\right] = J_{j-t},$$

$$E\left[v(t)e^{\mathrm{T}}(j)\right] = L_{j-t},$$

Then from the projection property, we have

$$\hat{w}(t \mid t + N) = \sum_{j=0}^{t+N} E\left[w(t)e^{\mathrm{T}}(j)\right] Q_e^{-1} e(j),$$

$$\hat{v}(t \mid t + N) = \sum_{j=0}^{t+N} E\left[v(t)e^{\mathrm{T}}(j)\right] Q_e^{-1} e(j).$$

Then thecolored noiseestimators (11)~(13) can be obtained.

IV. Lemmas[4]

Lemma 1

$$E\left[y(t)e^{\mathrm{T}}(j)\right] = P_{t-j} Q_e,$$

where

$$P_i = -A_1 P_{i-1} - \Lambda \Lambda - A_{n_a} P_{i-n_a} + D_i,$$

$$P_i = 0 \ (i < 0), \ D_i = 0 \ (i > n_d). \tag{14}$$

From Assumptions 2 we know

$$\mathrm{rank}\left[M^{\mathrm{T}} \quad H^{\mathrm{T}}\right]^{\mathrm{T}} = n.$$

Then there exists an $n \times m$ matrix T_0 such that $(M + T_0 H)$ is nonsingular. Premultiplying (2) by T_0 ,and combining it with (1) yield

$$x(t) = \Psi x(t-1) + \Psi_1 y(t) - \Psi_1 v(t) + \Psi_2 w(t-1), \tag{15}$$

where

$$\Psi = (M + T_0 H)^{-1} \Phi,$$
$$\Psi_1 = (M + T_0 H)^{-1} T_0,$$
$$\Psi_2 = (M + T_0 H)^{-1} \Gamma.$$

Lemma 2

System (1)~(2) have the completely observable pair (Ψ, H) under Assumptions 1~4.

Lemma 3

System (1)~(2) have the non-recursive state expression under Assumptions 1~4 as

$$x(t) = \sum_{i=0}^{\beta-1} \Omega_i [y(t+i) - \sum_{j=0}^{i-1} H \Psi^{i-1-j} [\Psi_1 y(t+j+1)$$

$$+ \Psi_1 v(t+j+1) + \Psi_2 w(t+j)] - v(t+i)], \quad (16)$$

where

$$\Omega = \begin{bmatrix} H \\ H\Psi \\ M \\ H\Psi^{\beta-1} \end{bmatrix},$$

$$\Omega^{\#} = (\Omega^{T}\Omega)^{-1}\Omega^{T} = [\Omega_0, \Omega_1, \Lambda, \Omega_{\beta-1}],$$

and we define

$$\Psi^i = 0 \ (i < 0), \ j \geq 0.$$

β is the observability index, satisfy

$$nm^{-1} \leq \beta \leq n - m + 1.$$

V. DESCRIPTOR STEADY STATE KALMAN ESTIMATORS

Theorem 2

Under Assumptions 1~4, the system (1)~(2) have the descriptor steady state Kalman estimator as

$$\hat{x}(t \mid t+N) = \Psi \hat{x}(t-1 \mid t-1+N) + \Psi_1 \hat{y}(t \mid t-1+N)$$

$$- \Psi_1 \hat{v}(t \mid t-1+N) + \Psi_2 \hat{w}(t-1 \mid t-1+N)$$

$$+ K_N e(t+N), \quad (17)$$

where

$$K_N = \sum_{i=0}^{\beta-1} \Omega_i \{ P_{i-N} - \sum_{j=0}^{i-1} H\Psi^{i-1-j} [\Psi_1 P_{j+1-N}$$

$$- \Psi_1 L_{N-j-1} Q_e^{-1} + \Psi_1 J_{N-j} Q_e^{-1}] - L_{N-i} Q_e^{-1} \}, \quad (18)$$

and in (18), we define

$$\Psi^i = 0 \ (i < 0), \ j \geq 0.$$

$\hat{w}(i \mid j)$ and $\hat{v}(i \mid j)$ can be computed by Theorem 1. $\hat{y}(i \mid j)$ can be computed recursively as

$$\hat{y}(i \mid j) = -A_1 \hat{y}(i-1 \mid j) - \Lambda \Lambda - A_{n_a} \hat{y}(i-n_a \mid j)$$

$$+ De(i), \quad (i > j), \quad (19)$$

where

$$e(k) = 0, \ (k > j),$$

and

$$\hat{y}(i \mid j) = y(i), \ (i \leq j).$$

Proof.

According to the projection property, we have

$$\hat{x}(t \mid t+N) = \hat{x}(t \mid t-1+N) + K_N e(t+N),$$

$$K_N = E[x(t) e^T (t+N)] Q_e^{-1}. \quad (20)$$

Taking the projection operation for (15) yields

$$\hat{x}(t \mid t-1+N) = \Psi \hat{x}(t-1 \mid t-1+N) + \Psi_1 \hat{y}(t \mid t-1+N)$$

$$- \Psi_1 \hat{v}(t \mid t-1+N) + \Psi_2 \hat{w}(t-1 \mid t-1+N) \quad (21)$$

Substituting (21) into (20) yields (17), Substituting (16), (10) and (14) into (20) yields (18).

Theorem 3

If Ψ is a stable matrix, then the descriptor steady-state Kalman estimators (17) are globally asymptotically stable, that is, $\hat{x}(t \mid t+N)$ are asymptotically independent of both the initial estimates $\hat{x}(t_0 \mid t_0 + N)$ and the innovation initial values $(e(0), \Lambda, e(n_d -1))$.

Proof.

The proof is similar to that as in [4], which is omitted.

Theorem 4

If Ψ is an unstable matrix, then for the system (1) and (2) under Assumptions 1~4, we can suitably select an $n \times m$ matrix $\overline{T}_0$, such that $\overline{\Psi} = \Psi + \overline{T}_0 H$ is stable, so that the pole assignment descriptor steady-state Kalman estimators have globally asymptotic stability as

$$\hat{x}(t \mid t+N) = \overline{\Psi} \hat{x}(t-1 \mid t-1+N) + \Psi_1 \hat{y}(t \mid t-1+N)$$

$$- \overline{T}_0 \hat{y}(t-1 \mid t-1+N) - \Psi_1 \hat{v}(t \mid t-1+N)$$

$$+ \overline{T}_0 \hat{v}(t-1 \mid t-1+N) + \Psi_2 \hat{w}(t-1 \mid t-1+N)$$

$$+K_N e(t+N) \qquad (22)$$

Proof.

The proof is similar to that as in [4] , which is omitted.

Note.

According to the principle of state observer[6], in order to ensure that the effect of the initial values $\hat{x}(t_0 \mid t_0 + N)$ is rapidlyforgotten, we usually assign the eigenvalues of $\overline{\Psi}$ values close to the origin. If we assign theeigenvalues of $\overline{\Psi}$ values close to the bound of the unit circle, then although we can make $\overline{\Psi}$ astable matrix, but the effect of the initial values $\hat{x}(t_0 \mid t_0 + N)$ will be forgotten in a longer decayingprocess.

Secondly, theorem 4 includes Theorem 3 as a special case.When Ψ is stable, we select $\overline{T}_0 = 0$. Theorem 4 can also be used to the case that Ψ is a stable matrix and its eigenvalues are approximate to the unitcircle. We can assign new eigenvalues of Ψ in order that the effect ofthe initial values $\hat{x}(t_0 \mid t_0 + N)$ can be rapidly forgotten.

VI. CONCLUSION

Using the modern time series analysis method, the steady-state Kalman estimator for descriptor system with colored noise is presented. It overcomes the limitations of previous estimators for only applying to white noise, and avoids the solution of the Riccatiequations. The algorithm is simple and easy to use. In order to obtain the global asymptotic stability of the estimator, the pole-assignment descriptor steady-state Kalman estimator with the global asymptotic stability is also presented by using the principle of the state observer. Compared with [1] and [2], it avoids computingthe optimal initial value of stateestimator. And the computational burden is reduced.Itismore suitable for real time applications.

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