Study of Vulnerability Parameters in a Class of Network

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Abstract. A communication network can be modeled by a graph, whose vertices represent the nodes and edges represent the links between sites of the network. Therefore, various problems in networks can be studied by graph theoretical methods. The vertex integrity, the edge-integrity not only give the minimum cost to disrupt the network, but also take into account what remains after destruction. In this paper, we give the explicit expressions for the two classes of integrity parameters of $K_n \times P_m$.

Introduction

A communication network can be modeled by a graph, whose vertices represent the nodes and edges represent the links between sites of the network. Therefore, various problems in networks can be studied by graph theoretical methods.

In an analysis of the stability of a network against disruption, there are generally two fundamental questions: (1) What is the size of the largest remaining group within which mutual communication can still occur? (2) How difficult is it to reconnect the network? The stability of a network can be measured by considering connectivity and edge-connectivity of the corresponding graph.

The connectivity and edge-connectivity give the minimum cost to disrupt the network, but they do not take into account what remains after destruction. In order to amending these weakness of two parameters, a number of vulnerability parameters have been introduced. Such as: (edge-) toughness, (edge-) integrity, scattering number, (edge-) tenacity, strength, rupture degree and several variations of (edge-) connectivity, in which the integrity includes the vertex-integrity, the edge-integrity, the weak-integrity, the pure edge-integrity.

The integrity, which takes into account the size of the largest component that remains after disconnection of the graph, toughness which takes into account the number of components once the graph has been disconnected, and tenacity which includes both the size of the largest component and the number of components remaining after disconnection of the graph [1, 2].

Formally, the vertex-integrity (frequently called just the integrity) is defined as:

$$I(G) = \min_{S \subset V(G)} \{| S | + m(G - s)\} \tag{1}$$

where *m(G - S)* denotes the number of vertices of the largest component of *G - S.* The edge-integrity

I(G) of a graph G is defined as

$$I'(G) = \min_{S \subseteq E(G)} \{| S | + m(G - s)\} \tag{2}$$

where $m(G - S)$ denotes the number of vertices of the largest component of $G - S$.

The vertex-integrity and edge-integrity were introduced by Barefoot et al. [3], and a number of basic results about them were established. Goddard was another person who has contributed greatly to the knowledge in the area of integrity in his doctoral dissertation [4]. Along with these existing results, K.S. Bagga et al. wrote two companion surveys [5, 6], devoted to the vertex-integrity and

edge-integrity. R. Laskar et al. [7] study the edge-integrity of some graphs and their complements, including some Nordhaus-Gaddum type results. Honest graphs, i.e. those which have the maximum possible edge-integrity, are also investigated. A. Vince [8] gave an upper bound on the integrity of any cubic graph with n vertices. A. Kirlangic and I. Buyukkuscu [9] calculated the integrity of double vertex graph of binomial trees B2, B3, and B4. There are results on the interrelations between integrity and other graph parameters [10], on the computational complexity of computing integrity [11] and results on 2-regular graphs [12]. The integrity and tenacity of generalized Petersen graphs and the relation with its invariant are considered by E. Kilic and P. Dundar [13].

It has been proved that computing most of these new vulnerability parameters are NP-hard. So, it is very important and interesting to study how to compute these parameters of special classed of graphs. In this paper, we main research on the vertex-integrity, edge-integrity of $K_n \times P_m$.

Main result

A few further comments on notation are appropriate here. Let G be a graph, $V(G)$ and $E(G)$ be the sets of vertices and edges of G, respectively. We denoted by Kn the complete graph with n vertices, and Pm (or Cm) a path (or a cycle) with m vertices, respectively. In this section, we will give the explicit expressions for the two classes of integrity parameters of $Kn \times Pm$.

Theorem 1 Let $n\ (\geq 3)$ and $m\ (\geq 2)$ be two integers, then we have

$$I(K_n \times P_m) = \left\lfloor \frac{m}{i+1} \right\rfloor \times n + ni, \tag{3}$$

where $i = \left\lceil \sqrt{m} - 1 \right\rceil$.

Proof: Suppose that S is a subset of $V(K_n \times P_m)$, such that $|S| + m(K_n \times P_m - S) = I(K_n \times P_m)$. According to the structure of the graph $K_n \times P_m$, we observe that every row is m copies of the complete graph K_n, and every column is n copies of the path P_m.

If we want to remove some vertices and make that $Kn \times Pm$ have at least two components, one must remove all vertices of a complete graph, which lead to the resulting graph have r components in such a way that in the previous $r-1$ components, every component will have i's complete graphs. In every $i + 1$ complete graphs, we remove the vertices of the last complete graph, so the number of vertices removed is $|s| = n \left\lfloor \frac{m}{i+1} \right\rfloor$ The largest component of $K_n \times P_m - S$ is any one component of the previous $r - 1$ components, and $m(K_n \times P_m - S) = ni$, then $I(K_n \times P_m) = \left\lfloor \frac{m}{i+1} \right\rfloor \times n + ni$.

It is clear that $I(K_n \times P_m) \leq \min\{\frac{m}{i+1}n + ni\}$ and $I(K_n \times P_m) \geq \min\{(\frac{m}{i+1} - 1)n + ni\}$. We define the function $f(i) = \frac{mn}{i+1} + ni = \frac{ni^2 + ni + mn}{i+1}$, the function $f(i)$ takes its minimum value at $f'(i) = \frac{ni^2 + 2ni + n - mn}{(i+1)^2} = 0$, then $i = \sqrt{m} - 1$, by computation, we can obtain $i = \left\lceil \sqrt{m} - 1 \right\rceil$.

Similar to the function $g(i) = (\frac{m}{i+1} - 1)n + ni = \frac{m^2 mn - n}{i+1}$, and $g(i)$ takes its minimum value at $g'(i) = \frac{ni^2 + 2ni + n - mn}{(i+1)^2} = 0$, then $i = \left\lceil \sqrt{m} - 1 \right\rceil$, by computation, i is the same value as the above case.

So we have $I(Kn \times Pm) = \left\lfloor \frac{m}{i+1} \right\rfloor \times n + ni$, and $i = \left\lceil \sqrt{m} - 1 \right\rceil$.

Theorem 2 Let $n\ (\geq 2)$ and $m\ (\geq 2)$ be two integers,

$$I'(K_n \times P_m) = n \left\lfloor \frac{m-1}{i} \right\rfloor + ni, \tag{4}$$

where $i = \left\lceil \sqrt{m} - 1 \right\rceil$.

Proof: Suppose that S is a subset of $(K_n \times P_m)$, such that $|S| + m(K_n \times P_m - S) = I'(K_n \times P_m)$. According to the structure of the graph $K_n \times P_m$ and the definition of the edge-integrity, if we want

to remove some edges of $K_n \times P_m$ and make that the resulting graph have at least two components, one must remove all edges between two complete graphs.

Without loss of generality, we assume that the resulting graph have r components in such a way that every component in the previous $r - 1$ components have i's complete graphs (except for the rth component). We can delete all edges between the ith complete graph and the $i + 1$th complete graph in every i complete graphs, the number of edges that we deleted is $/S/ = n\lfloor \frac{m-1}{i} \rfloor$. The largest component of $K_n \times P_m - S$ is any one component of the previous $r - 1$ components, so $m(K_n \times P_m - S) = ni$, and we have $I'(K_n \times P_m) = \lfloor \frac{m-1}{i} \rfloor n + ni$.

Now we choose an suitable value of i such that the value of $\lfloor \frac{m-1}{i} \rfloor n + ni$ is minimum. With the same proof of Theorem 1, $I'(K_n \times P_m) \le \min\{\frac{m-1}{i}n + ni\}$, and $I(K_n \times P_m) \ge \min\{(\frac{m-1}{i}-1)n + ni\}$.

Define the function $f(i) = \frac{(m-1)n}{i} + ni = \frac{m^2+mn-n}{i}$, the function $f(i)$ takes its minimum value at $f'(i) = \frac{nt^2+n-mn}{i^2} = 0$, then $i = \sqrt{m-1}$, by computation, we find $i = \lceil \sqrt{m-1} \rceil$.

Similar to the function $g(i) = (\frac{m-1}{i}-1)n + ni = \frac{m^2-nt+mn-n}{i}$, the function $g(i)$ takes its minimum value at $g'(i) = \frac{nt^2+n-mn}{i^2} = 0$, then $i = \sqrt{m-1}$, by computation, we find $i = \lceil \sqrt{m-1} \rceil$.

So we have $I'(K_n \times P_m) = \lfloor \frac{m-1}{i} \rfloor n + ni$ and $i = \lceil \sqrt{m-1} \rceil$.

Conclusions

A communication network can be modeled by a graph, whose vertices represent the nodes and edges represent the links between sites of the network. The stability of a network can be measured by considering some vulnerability parameters of the corresponding graph. In this paper, we give the explicit expressions for the two classes of integrity parameters of $K_n \times P_m$.

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