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Characterizations of Distributions Via Conditional Expectation of Function of Generalized Order Statistics

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Characterizations of probability distributions by different regression conditions on generalized order statistics have attracted the attention of many researchers. We present here, characterization of distributions based on the conditional expectation of generalized order statistics extending the characterization results reported by Noor and Athar (2014).

Keywords: Conditional expectation; generalized order statistics; upper record values.

Mathematics Subject Classification: 62E10, 60E05, 60E15

1. Introduction

The concept of generalized order statistics (*gos*) was introduced by Kamps [1] in terms of their joint *pdf* (probability density function). The order statistics, record values, k-record values, Pfifer records and progressive type II order statistics are special cases of the *gos*. The *rv's* (random variables) $X(1,n,m,k)$, $X(2,n,m,k)$,..., $X(n,n,m,k)$, $k>0$, $m\in\mathbb{R}$, are n *gos* from an absolutely continuous *cdf* (cumulative distribution function) F with corresponding *pdf* f if their joint *pdf*, $f_{1,2,\ldots,n}\left(x_1,x_2,\ldots,x_n\right)$, can be written as

$$f_{1,2,\ldots,n}(x_1,x_2,\ldots,x_n) = k\left(\Pi_{j=1}^{n-1}\gamma_j\right)\left[\Pi_{j=1}^{n-1}\left(\overline{F}(x_j)\right)^m f(x_j)\right]\times\left(\overline{F}(x_n)\right)^{k-1} f(x_n),$$
$$F^{-1}(0+) < x_1 < x_2 < \cdots < x_n < F^{-1}(1-), \tag{1.1}$$

where $\overline{F}(x) = 1-F(x)$ and $\gamma_j = k+(n-j)(m+1)$ for all j, $1 \le j \le n$, k is a positive integer and $m \ge -1$.

If $k = 1$ and $m = 0$, then $X(r,n,m,k)$ reduces to the ordinary rth order statistic and (1.1) will be the joint *pdf* of order statistics $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ from F. If $k = 1$ and $m = -1$, then (1.1) will be the joint *pdf* of the first n upper record values of the *i.i.d.* (independent and identically distributed) *rv's* with *cdf* F and *pdf* f.

Integrating out $x_1,x_2,\ldots,x_{r-1},x_{r+1},\ldots,x_n$ from (1.1) we obtain the *pdf* $f_{r,n,m,k}$ of $X(r,n,m,k)$

$$f_{r,n,m,k}(x) = \frac{c_{r_{-1}}}{(r-1)!}\left(\overline{F}(x)\right)^{\gamma_r-1} f(x)\, g_m^{r-1}(F(x)), \tag{1.2}$$

where $c_{r_{-1}} = \Pi_{j=1}^{r}\ \gamma_j$ and

$$g_m(x) = h_m(x) - h_m(0) = \frac{1}{m+1}\left[1-(1-x)^{m+1}\right], \quad m \ne -1,$$
$$= -\ln(1-x), \qquad m = -1, \quad x \in (0,1),$$

and

$$h_m(x) = -\frac{1}{m+1}(1-x)^{m+1}, \quad m \ne -1,$$
$$= -\ln(1-x), \qquad m = -1, \quad x \in (0,1).$$

Note that, since $\lim_{m\to -1}\frac{1}{m+1}\left[1-(1-x)^{m+1}\right] = -\ln(1-x)$, we will write

$$g_m(x) = \frac{1}{m+1}\left[1-(1-x)^{m+1}\right], \text{ for all } x \in (0,1) \text{ and all } m \text{ with}$$
$$g_{-1}(x) = \lim_{m\to -1} g_m(x).$$

The joint *pdf* of $X(s,n,m,k)$ and $X(r,n,m,k)$, $r < s$, is given by (see Kamps [1], p.68)

$$f_{s,r,n,m,k}(x,y) = \tfrac{c_{s-1}}{(r-1)!(s-r-1)!}\left(\overline{F}(x)\right)^{m-1} f(x)\, g_m^{r-1}(F(x)) \times$$
$$\left[h_m(F(y)) - h_m(F(x))\right]^{s-r-1}\left(\overline{F}(y)\right)^{\gamma_s-1} f(y), \quad y \ge x.$$

Consequently, the conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, for $m \ne -1$, is

$$f_{s\,|\,r,n,m,k}(y\,|\,x) = \frac{c_{s-1}}{c_{r-1}(s-r-1)!}[h_m(F(y)) - h_m(F(x))]^{s-r-1}\frac{(\overline{F}(y))^{\gamma_s-1}}{(\overline{F}(x))^{\gamma_{r+1}}} f(y), \quad y > x. \tag{1.3}$$

Following Noor and Athar [2], the indices at which the upper record values occur are called the record times $\{U(n)\}$, $n > 0$, where $U(n) = mim\left\{j \,|\, j > U(n-1), X_j > X_{U(n-1)}, n > 1\right\}$ and $U(1) = 1$. Noor and Athar [2] prove the following Theorems 1.1 and 1.2.

Theorem 1.1. *Let X be an absolutely continuous random variable with cdf $F(x)$ and pdf $f(x)$ on the support (α, β). Then, for two values of r and s, $1 \le r < s \le n$,*

$$E\left[\left\{h\left(X_{U(s)}\right) - h\left(X_{U(r)}\right)\right\}^p | X_{U(r)} = x\right] = g_{r,s,p} = \frac{1}{a^p}\frac{\Gamma(p+s-r)}{\Gamma(s-r)},$$

if and only if

$$F(x) = 1 - e^{-ah(x)}, \quad a > 0,$$

where $h(x)$ is a continuous, differentiable, and non-decreasing function of x and p is a positive integer.

Theorem 1.2. *Let X be an absolutely continuous random variable with cdf $F(x)$ and pdf $f(x)$ on the support (α, β). Then, for two consecutive values of r and s, $1 \le r < s \le n$,*

$$E\left[\left\{h\left(X_{U(s)}\right) - h\left(X_{U(r)}\right)\right\}^p | X_{U(r)} = x\right] = a^* \sum_{j=0}^{p} \binom{p}{j} (h(x))^{p-j} \left(\frac{b}{a}\right)^j,$$

if and only if

$$\overline{F}(x) = [ah(x) + b]^c, \quad a \neq 0,$$

where

$$a^* = \sum_{i=0}^{p} (-1)^{i+p} \binom{p}{j} \left(\frac{c}{c+i}\right)^{s-r}$$

and $h(x)$ is a continuous and differentiable function of x .

2. Characterization Results

We present here, characterization of distributions based on the conditional expectation of generalized order statistics extending the characterization results reported by Noor and Athar [2].

Theorem 2.1. *Let the random variable $X : \Omega \to (0,\infty)$ have an absolutely continuous* cdf $F(x)$ *with* pdf $f(x)$. *The following two conditions are equivalent:*

(a) $F(x) = 1 - e^{-q(x)}, q(x)$ *is differentiable, increasing and* $\lim_{x\to\theta} q(x) = \infty$.

(b) $E[\{q\{X(s,n,m,k)) - q(X(x,n,m,k))\}^p \,|X(r,n,m,k) = x]$
$$= \frac{\gamma_{r+1}\cdots\gamma_s}{(m+1)^{s-r-1}} \sum_{j=0}^{s-r-1} (-1)^j \frac{1}{\Gamma(j+1)\Gamma(s-r-j)} \frac{\Gamma(p+1)}{(\gamma_{s-j})^{p+1}},$$
for two consecutive $s-1,\ s,\ 1 < r+1 < s \le n$, *and* $p \in \mathbb{N}$.

Proof. Assume (a) holds, then we have

$$\begin{aligned}
I &= E[\{q\{X(s,n,m,k)) - q(X(r,n,m,k))\}^p \,|X(r,n,m,k) = x] \\
&= \frac{c_{s-1}}{c_{r-1}(s-r-1)!} \int_x^\infty (q(y)-q(x))^p [h_m(F(y)) - h_m(F(x))]^{s-r-1} \frac{(\overline{F}(y))^{\gamma_s-1}}{(\overline{F}(x))^{\gamma_{r+1}}} f(y)dy \\
&= \frac{c_{s-1}}{c_{r-1}(m+1)^{s-r-1}} \int_x^\infty \sum_{j=o}^{s-r-1} (-1)^j \frac{(q(y)-q(x))^p}{\Gamma(j+1)\Gamma(s-r)} e^{-\gamma_{s-j}(q(y)-q(x))} q\prime(y)dy.
\end{aligned}$$

Substituting $q(y) - q(x) = t$, we arrive at

$$\begin{aligned}
I &= \frac{c_{s-1}}{c_{r-1}(m+1)^{s-r-1}} \int_0^\infty \sum_{j=o}^{s-r-1} \frac{(-1)^j}{\Gamma(j+1)\Gamma(s-r-j)} t^p e^{-\gamma_{s-j}} dt \\
&= \frac{\gamma_{r+1}\cdots\gamma_s}{(m+1)^{s-r-1}} \sum_{j=o}^{s-r-1} \frac{(-1)^j}{\Gamma(j+1)\Gamma(s-r-j)} \frac{\Gamma(p+1)}{(\gamma_{s-j})^{p+1}},
\end{aligned}$$

which is (b).

Now, assume (b) holds, then observe that

$$\begin{aligned}
&\int_x^\infty (q(y)-q(x))^p [h_m(F(y)) - h_m(F(x))]^{s-r-1} (\overline{F}(y))^{\gamma_s-1} f(y)dy \\
&\quad = \frac{1}{(m+1)^{s-r-1}} \sum_{j=0}^{s-r-1} (-1)^j \frac{\Gamma(s-r)}{\Gamma(j+1)\Gamma(s-r-j)} \frac{\Gamma(p+1)}{(\gamma_{s-j})^{p+1}} (\overline{F}(x))^{\gamma_{r+1}}.
\end{aligned}$$

Differentiating both sides of the above equation with respect to x, we obtain

$$\begin{aligned}
&-pq'(x) \int_x^\infty (q(y)-q(x))^{p-1} [h_m(F(y)) - h_m(F(x))]^{s-r-1} (\overline{F}(y))^{\gamma_s-1} f(y)\,dy \\
&-(s-r-1)(\overline{F}(x))^m f(x) \int_x^\infty (q(y)-q(x))^p [[h_m(F(y)) - h_m(F(x))]^{s-r-2} (\overline{F}(y))^{\gamma_s-1} f(y)\,dy \\
&\quad = -\frac{1}{(m+1)^{s-r-1}} \frac{f(x)}{\overline{F}(x))} \sum_{j=0}^{s-r-1} (-1)^j \frac{\Gamma(s-r)}{\Gamma(j+1)\Gamma(s-r-j)} \frac{\Gamma(p+1)}{(\gamma_{s-j})^{p+1}} (\overline{F}(x))^{\gamma_{s+1}}.
\end{aligned}$$

In view of (b) and the above equation, we have

$$-\frac{pq\prime(x)}{(m+1)^{s-r-1}}\sum_{j=o}^{s-r-1}\frac{\Gamma(s-r)(-1)^j}{\Gamma(j+1)\Gamma(s-r-j)}\frac{\Gamma(p)}{(\gamma_{s-j})^p}(\overline{F}(x))^{\gamma_{s+1}}$$
$$-\frac{1}{(m+1)^{s-r-1}}\frac{f(x)}{\overline{F}(x))}(\overline{F}(x))^{m+1}\sum_{j=o}^{s-r-2}\frac{(-1)^j\Gamma(s-r)(-1)^j(m+1)}{\Gamma(j+1)\Gamma(s-r-j-1)}\frac{\Gamma(p)}{(\gamma_{s-j})^p+1}(\overline{F}(x))^{\gamma_{s+2}}$$
$$=-\frac{\gamma_{r+1}}{(m+1)^{s-r-2}}\frac{f(x)}{\overline{F}(x))}\sum_{j=o}^{s-r-1}\frac{(-1)^j\Gamma(s-r))}{\Gamma(j+1)\Gamma(s-r-j)}\frac{\Gamma(p)}{(\gamma_{s-j})^p}(\overline{F}(x))^{\gamma_{s+1}}.$$

Further,

$$-\frac{q\prime(x)}{(m+1)^{s-r-1}-}\sum_{j=o}^{s-r-2}\frac{(-1)^j\Gamma(s-r))}{\Gamma(j+1)\Gamma(s-r)}\frac{\Gamma(p+1)}{(\gamma_{s-j})^p}$$
$$-q'(x)(-1)^{s-r-1}\frac{\Gamma(s-r)}{\Gamma(s-r)}\frac{\Gamma(p+1)}{(\gamma_{r+1})^p}$$
$$=\frac{f(x)}{\overline{F}(x))}\frac{1}{(*m+1)^{s-r-1}}\sum_{j=o}^{s-r-2}\frac{(-1)^{j\Gamma}\Gamma(s-r))}{\Gamma(j+1)\Gamma(s-r)}\frac{(s-r-1-j)(m+1)}{(m+1)s-r-1-i}\frac{\Gamma(p+1)}{(\gamma_{s-j})^{p+1}}$$
$$-\gamma_{r+1}\frac{f(x)}{\overline{F}(x))}\sum_{j=o}^{s-r-2}\frac{(-1)^j}{\Gamma(j+1)\Gamma(s-r)}\frac{\Gamma(p+1)}{(\gamma_{s-j})^p}$$
$$-\gamma_{r+1}\frac{f(x)}{\overline{F}(x))}(-1)^{s-r-1}\frac{\Gamma(p+1)}{(\gamma_{r+1j})^{p+1}}.$$

i.e.

$$-\frac{q\prime(x)}{(m+1)^{s-r-1}-}\sum_{j=o}^{s-r-2}\frac{(-1)^j\Gamma(s-r))}{\Gamma(j+1)\Gamma(s-r)}\frac{\Gamma(p+1)}{(\gamma_{s-j})^p}-q'(x)(-1)^{s-r-1}\frac{\Gamma(s-r)}{\Gamma(s-r)}\frac{\Gamma(p+1)}{(\gamma_{r+1})^p}$$
$$=-\frac{f(x)}{\overline{F}(x))}\frac{1}{(*m+1)^{s-r-1}}\sum_{j=o}^{s-r-2}\frac{(-1)^j\Gamma(s-r))}{\Gamma(j+1)\Gamma(s-r)}\frac{(s-r-1-j)(m+1)}{(m+1)s-r-1-i}\frac{\Gamma(p+1)}{(\gamma_{s-j})^{p1}}$$
$$-\frac{f(x)}{\overline{F}(x))}(-1)^{s-r-1}\frac{\Gamma(p+1)}{(\gamma_{r+1j})^p},$$

from which we obtain

$$\frac{f(x)}{\overline{F}(x))}=q(x).$$

Upon integrating, we arrive at $\overline{F}(x)=ce^{-g(x)}$, where c is a constant. Using the boundary conditions $\overline{F}(0)=1$ and $\lim_{x\to\infty}\overline{F}(x)=0$, we must have $\overline{F}(x)=e^{-q(x)}$. □

Remark 2.1. For $m=-1$, Theorem 2.1 reduces to Theorem 1.1.

The following Theorem is a generalization of Theorem 1.2.

Theorem 2.2. *Let the random variable* $X:\Omega\to(0,\infty)$ *have an absolutely continuous* cdf $F(x)$ *with* pdf $f(x)$. *The following two conditions are equivalent:*

(a) $F(x)=1-q(x), 0<q(x)<1,$ *and* $q(x)$ *is diffirentiable and decreasing.*

(b)
$$E\left[\{q\{X(r,n,m,k))-q(X(s,n,m,k))\}^p\,|X(r,n,m,k)=x\right]$$
$$=\frac{\gamma_{r+1}\cdots\gamma_s}{(m+1)^{s-r-1}}(q(x))^p\sum_{j=0}^{s-r-1}(-1)^j\frac{1}{\Gamma(j+1)\Gamma(s-r-j)}\frac{\Gamma(p+1)\Gamma(c_{s-j})}{\Gamma(p+1+c\gamma_{s-j})},$$
for two conecuetive r *and* $r+1$, $1\le r+1<s\le n$ *and* $p\in\mathbb{N}$.

Proof. Assume (a) holds, then we have

$$\begin{aligned}I=&E\left[\{q\{X(r,n,m,k))-q(X(s,n,m,k))\}^p\,|X(r,n,m,k)=x\right]\\&=\frac{c_{s-1}}{c_{r-1}(s-r-1)!}\int_x^\infty(q(x)-q(y))^p[h_m(F(y))-h_m(F(x))]^{s-r-1}\frac{(\overline{F}(y))^{\gamma_s-1}}{(\overline{F}(x))^{\gamma_{r+1}}}f(y)dy\\&=\frac{c_{s-1}}{c_{r-1}(m+1)^{s-r-1}}\int_x^\infty\sum_{j=o}^{s-r-1}(-1)^j\frac{(q(x)-q(y))^p}{\Gamma(j+1)\Gamma(s-r)}\frac{(q(y))^{\gamma_{s-j-1}}}{(q(x))^{\gamma_{s-j}}}q\prime(y)dy.\end{aligned}$$

Substituting $\frac{q(y)}{q(x)}=t$, we have

$$\begin{aligned}I=&\frac{\gamma_{r+1}\cdots\gamma_s}{(m+1)^{s-r-1}}(q(x))^p\sum_{j=0}^{s-r-1}\frac{(-1)^j}{\Gamma(j+1)\Gamma(s-r)}\int_o^1t^p(1-t)t^{\gamma_{s-j-1}}dt\\&=\frac{\gamma_{r+1}\cdots\gamma_s}{(m+1)^{s-r-1}}(q(x))^p\sum_{j=0}^{s-r-1}\frac{(-1)^j}{\Gamma(j+1)\Gamma(s-r)}\frac{\Gamma(p+1)\Gamma(c_{s-j})}{\Gamma(p+1+c\gamma_{s-j})},\end{aligned}$$

which is (b).

Now, assume (b) holds, then we have

$$\begin{aligned}I\prime_{s,r,n,m,k.p}&=E\left[\{q(X(r,n,m,k))-q(X(s,n,m,k))\}^p\,|X(r,n,m,k)=x\right]\\&=\frac{c_{s-1}}{c_{r-1}(s-r-1)!}\int_x^\beta(q(x)-q(y))^p[h_m(F(y))-h_m(F(x))]^{s-r-1}\frac{(\overline{F}(y))^{\gamma_s-1}}{(\overline{F}(x))^{\gamma_{r+1}}}f(y)dy.\end{aligned}$$

Differentiating both side of the above equation with respect to x, we obtain

$$
\begin{aligned}
&\frac{d}{dx} I_{s\,|\,r,n,m,k,p} \\
&\quad = \frac{p c_{s-1} q'(x)}{c_{r-1}(s-r-1)!} \int_x^{\infty} (q(x)-q(y))^{p-1} [h_m(F(y)) - h_m(F(x))]^{s-r-1} \frac{(\overline{F}(y))^{\gamma_s - 1}}{(\overline{F}(x))^{\gamma_{r+1}}} f(y) dy \\
&\quad - \frac{c_{s-1}(s-r-1)}{c_{r-1}(s-r-2)!} (1-F(x))^m f(x) \int_x^{\infty} (q(x)-q(y))^{p} [h_m(F(y)) - h_m(F(x))]^{s-r-2} \times \\
&\quad \frac{(\overline{F}(y))^{\gamma_s - 1}}{(\overline{F}(x))^{\gamma_{r+1}}} f(y) dy - \gamma_{r+1} \frac{p c_{s-1}}{c_{r-1}(s-r-1)!} \frac{f(x)}{1-F(x)} \int_x^{\infty} (g(x)-g(y))^{p} \times \\
&\quad [h_m(F(y)) - h_m(F(x))]^{s-r-1} \frac{(\overline{F}(y))^{\gamma_s - 1}}{(\overline{F}(x))^{\gamma_{r+1}}} f(y) dy,
\end{aligned}
$$

i.e.

$$
\gamma_{r+1} \frac{f(x)}{1-F(x)} = \frac{\frac{d}{dx} I_{s\,|\,r,n,m,k,p} - p q'(x) I_{s,r.n.m.k.p-1}}{\gamma_{r+1} I_{s,r,n,m,k,p} - I_{s,r+1,n,m,k.p}}.
$$

Now

$$
\begin{aligned}
&fracddx I_{s,r,n,m,k.p} - p q'(x) I_{s,r,n,m,k.p-1} \\
&\quad = \frac{d}{dx} \Big(\frac{\gamma_{r+1} \ldots . \gamma_s}{(m+1)^{s-r-1}} (q(x))^p \sum_{j=0}^{s-r-1} \frac{(-1)^j}{\Gamma(j+1)\Gamma(s-r-j)} \frac{\Gamma(p+1)\Gamma(c_{s-j})}{\Gamma(p+1+c\gamma_{s-j})} \\
&\quad - p q'(x) \frac{\gamma_{r+1} \ldots . \gamma_s}{(m+1)^{s-r-1}} (q(x))^{p-1} \sum_{j=0}^{s-r-1} \frac{(-1)^i}{\Gamma(j+1)\Gamma(s-r-j)} \frac{\Gamma(p)\Gamma(\gamma_{s-j})}{\Gamma(p+c\gamma_{s-j})} \\
&\quad = \frac{\gamma_{r+1} \ldots . \gamma_s}{(m+1)^{s-r-1}} p (q(x))^{p-1} q'(x) \sum_{j=0}^{s-r-1} \frac{(-1)^{s-r-1}}{\Gamma(j+1)\Gamma(s-r-j)} \frac{\Gamma(p+1)\Gamma(\gamma_{s-j})}{\Gamma(p+1+\gamma_{s-j})} \\
&\quad - q'(x) \frac{\gamma_{r+1} \ldots . \gamma_s}{(m+1)^{s-r-1}} (q(x))^{p-1} \sum_{j=0}^{s-r-1} \frac{(-1)^{s-r-1}}{\Gamma(j+1)\Gamma(s-r-j)} (p+\gamma_{s-j}) \frac{\Gamma(p+1)\Gamma(\gamma_{s-j})}{\Gamma(p+1++\gamma_{s-j})} \\
&\quad = - \frac{\gamma_{r+1} \ldots . \gamma_s}{(m+1)^{s-r-1}} (q(x))^{p-1} q'(x) \sum_{j=0}^{s-r-2} (-1)^{s-r-1} \frac{\gamma_{s-j} \Gamma(p+1)\Gamma(\gamma_{s-j})}{\Gamma(p+1+c\gamma_{s-j})} \\
&\quad - \frac{\gamma_{r+1} \ldots . \gamma_s}{(m+1)^{s-r-1}} (q(x))^{p-1} q'(x) \frac{(-1)^{s-r-2}}{\Gamma(j+1)\Gamma(s-r-j)} \frac{\gamma_{s-j} \Gamma(p+1)\Gamma(\gamma_{r+1})}{\Gamma(p+1+c\gamma_{r+1})}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&I_{s,r,n,m,k,p} - I_{s,r+1,n,m,k.p} \\
&\quad = \frac{\gamma_{r+1}\cdots\gamma_s}{(m+1)^{s-r-1}}(q(x))^p q'(x)\sum_{j=0}^{s-r-1}\frac{(-1)^j}{\Gamma(j+1)\Gamma(s-r-j)}\frac{\Gamma(p+1)\Gamma(\gamma_{s-j})}{\Gamma(p+1+c\gamma_{s-j})} \\
&\quad\quad - \frac{\gamma_{r+2}\cdots\gamma_s}{(m+1)^{s-r-1}}(q(x))^p q'(x)\sum_{j=0}^{s-r-2}\frac{(-1)^j}{\Gamma(j+1)\Gamma(s-r-j-1)}\frac{\Gamma(p+1)\Gamma(\gamma_{s-j})}{\Gamma(p+1+c\gamma_{s-j})} \\
&\quad = \frac{\gamma_{r+1}\cdots\gamma_s}{(m+1)^{s-r-1}}(q(x))^p q'(x)\sum_{j=0}^{s-r-1}\frac{(-1)^j}{\Gamma(j+1)\Gamma(s-r-j-1)}\frac{\Gamma(p+1)\Gamma(\gamma_{s-j}))}{\Gamma(p+1+c\gamma_{s-j})} \\
&\quad\quad - \frac{\gamma_{r+2}\cdots\gamma_s}{(m+1)^{s-r-1}}(q(x))^p q'(x)\sum_{j=0}^{s-r-2}\frac{(-1)^{s-r-1}(m+1)(s-r-j-1)}{\Gamma(j+1)\Gamma(s-r-1)}\frac{\Gamma(p+1)\Gamma(\gamma_{s-j})}{\Gamma(p+1+c\gamma_{s-j})} \\
&\quad = \frac{\gamma_{r+2}\cdots\gamma_s}{(m+1)^{s-r-1}}(q(x))^p q'(x)\sum_{j=0}^{s-r-2}\frac{(-1)^j\gamma_{s-j}}{\Gamma(j+1)\Gamma(s-r-1)}\frac{\Gamma(p+1)\Gamma(\gamma_{s-j})}{\Gamma(p+1+c\gamma_{s-j})} \\
&\quad\quad + \frac{\gamma_{r+2}\cdots\gamma_s}{(m+1)^{s-r-1}}(q(x))^p q'(x)\frac{(-1)^j\gamma_{r+1}}{\Gamma(j+1)\Gamma(s-r-1)}\frac{\Gamma(p+1)\Gamma(\gamma_{r+1})}{\Gamma(p+1+c\gamma_{r+1})}.
\end{aligned}
$$

Thus

$$
\gamma_{r+1}\frac{f(x)}{1-F(x)} = \frac{\frac{d}{dx}I_{s\,|\,r,n,m,k,p} - pq'(x)I_{s,r.n.m.k.p-1}}{\gamma_{r+1}I_{s,r,n,m,k,p} - I_{s,r+1,n,m,k.p}} = -\frac{\gamma_{r+1}q'(x)}{q(x)},
$$

i.e.

$$
1 - F(x) = cq(x).
$$

Using the boundary conditions $\lim_{x\to 0} q(x) = 1$, we must have $c = 1$, completing the proof. □

Remark 2.2. We can take $q(x) = [ah(x)+b]^c$ in Theorem 2.2.

Milwaska and Szynal [3], define the kth upper record value of $X_j's$ by $Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1}$. They assume that the common random variable X is an absolutely continuous random variable concentrated on the interval (α,β), with $F(x) < 1$ for $x \in (\alpha,\beta)$, $F(\alpha) = 0$ and $F(\beta) = 1$. For a given monotonic and differentiable function ϕ on (α,β), they write

$$
\overline{\mu}^{(k)}_{m+1\,|\,m} = E\left[\phi\left(Y^{(k)}_{m+1}\right)|\,Y^{(k)}_m = x\right],
$$
$$
\text{and } \overline{\mu}^{(k)}_{m\,|\,m+1} = E\left[\phi\left(Y^{(k)}_{m}\right)|\,Y^{(k)}_{m+1} = y\right],
$$

and prove the following characterizations of the family of distributions $F(x) = 1 - [a\phi(x)+b]^c$ (Theorems 2.3 and 2.4 below), among other characterizations of families of distributions.

Theorem 2.3. *If k is a positive integer,*

$$1-F(x)=[a\phi(x)+b]^{c},$$

if and only if

$$\overline{\mu}_{m+1|m}^{(k)}=\frac{1}{kc+1}\left[kc\phi(x)-\frac{b}{a}\right],$$

where $a\neq 0$, $b,c>0$ are finite constants.

Theorem 2.4. *If k is a positive integer,*

$$1-F(x)=[a\phi(x)+b]^{c},$$

if and only if

$$\overline{\mu}_{m+1|m+2}^{(k)}=-\phi(y)\frac{c(m+1)}{\overline{H}(y)}+\frac{c(m+1)}{\overline{H}(y)}\overline{\mu}_{m|m+1}^{(k)}-\frac{b}{a},$$

and

$$\overline{\mu}_{1|2}^{(k)}=\frac{c[\phi(\alpha)-\phi(y)]}{\overline{H}(y)}-\frac{b}{a},$$

where $\overline{H}(y)=-\log[1-F(y)]$ and $a\neq 0$, $b,c\neq 0$ are finite constants.

Hamedani et al. [4] presented the following characterization result, among others, in the spirit of the above two characterizations.

Theorem 2.5. $1-F(x)=[a\phi(x)+b]^{c}$ *if and only if*

$$E[\phi(X)\,|\,X\geq x]=\frac{1}{c+1}\left[c\phi(x)-\frac{b}{a}\right],\quad \alpha<x<\beta,$$

where $a\neq 0, b,c>0$ are finite constants.

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