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On Hilbert C^*-module-valued Random Variables

K. Shafie
Applied Statistics and research Methods,University of Northern Colorado
Greeley, 80639 Colorado
khalil.shafie@unco.edu

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In this paper random variables that take their values from a Hilbert C^*-module are defined and three definitions for the mean, covariance operator, and Gaussian distribution of these random variables are given and it is shown that these definitions are equivalent. Furthermore, the concept of covariance of two real valued random variables and its properties are extended to two Hilbert C^*-module valued random variables. These lead us to the generalization of Rao-Blackwell theorem for this type of random variables. Finally, in a special case, it is proved that the finiteness of second moment of the norm of such a random variable is a sufficient condition for the central limit theorem to be true.

Keywords: Banach Valued random variable; central limit theorem, Covariance operator, Hilbert C^* modules.

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1. Introduction

In some studies we have a two dimensional vector whose components belong to a Hilbert space $\mathbb{H}$ with the inner product $<,.,>_{\mathbb{H}}$. For example when we have vector valued random variable whose components are members of $L^2(\Omega, P)$, for some probability space $(\Omega, \mathscr{F}, P)$. Another example is in the functional data analysis where we have a two dimensional vector whose elements are square integrable functions on the real line (see Ramsay and Silverman [13]). In these cases if x_1, y_1, x_2 and y_2 belong to $\mathbb{H}$ then $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ belong to the product space $\mathbb{H} \times \mathbb{H}$ and the ordinary inner product on the product space is defined by

$$< z_1, z_2 > = < x_1, x_2 >_{\mathbb{H}} + < y_1, y_2 >_{\mathbb{H}} .$$

By this definition inner product in the product space always results in a scalar value and two vectors $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are orthogonal if $< x_1, x_2 >=< y_1, y_2 >= 0$ or $< x_1, x_2 >= - < y_1, y_2 >$. At least from statistical point of view, it does not make sense to call two vectors orthogonal when the second situation happens. An alternative way to define an inner product on $\mathbb{H} \times \mathbb{H}$ is:

$$< z_1, z_2 >= \begin{bmatrix} < x_1, x_2 >_{\mathbb{H}} & < x_1, y_2 >_{\mathbb{H}} \\ < y_1, x_2 >_{\mathbb{H}} & < y_1, y_2 >_{\mathbb{H}} \end{bmatrix}.$$

This inner product along with the multiplication of vectors in $\mathbb{H} \times \mathbb{H}$ by 2×2 matrices from left makes $\mathbb{H} \times \mathbb{H}$ a Hilbert C^*-module. These spaces introduced by Kaplansky [5] and studied by

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Paschke [11], today are a very useful tool in many areas of mathematics and physics. Lance [6] is a good reference on the theory of Hilbert C^*-modules. For the use of Hilbert C^*-module tools in stochastic processes see Popovich [12] and for the relationship between the wavelet theory and Hilbert C^*-module see Packer and Rieffel [9, 10].

Study of random variables with values in a Hilbert C^*-module, from the applied and theoretical point of view seems to be very useful. Although these random variables are a special case of Banach-valued random variables but since they have a structure similar to Hilbert-valued random variables they deserve an independent study. In this paper we will study some elementary properties of Hilbert C^*-module valued random variables. Note that the space of linear operators from a linear space $\mathbb{A}$ to another linear space $\mathbb{B}$ will be denoted by $L(\mathbb{A},\mathbb{B})$, the transpose of a vector a by a' and the involution of an element a in a C^* algebra by a^*.

The paper is organized as follows. In Section 2 a short review of basic concept of Hilbert C^*-module spaces is given. Then in Section 3 we have a brief review of properties of Banach-valued random variables. In Section 4 the main contribution of the paper is given that include the definition of Hilbert C^*-module random variables, their mean, covariance operator and Gaussian distribution. Furthermore, in Section 5 the concept of covariance of two real valued random variables and its properties are extended to two Hilbert C^*-module valued random variables.

2. Hilbert C^*-module spaces

In this section using Lance [6] we will have a short review on Hilbert C^*-module spaces and give some examples of these spaces.

Let $\mathbb{A}$ be a C^* algebra with a unit element and $\mathbb{H}^*$ be a linear space on the field of real numbers $\mathbb{R}$. Suppose $\mathbb{H}^*$ is a left $\mathbb{A}$ module and for $h \in \mathbb{H}^*$, $\lambda \in \mathbb{R}$ and $a \in \mathbb{A}$ we have $\lambda(ah) = a(\lambda h)$. An $\mathbb{A}$-valued inner product on $\mathbb{H}^*$ is an operator $< .,. >$: $\mathbb{H}^* \times \mathbb{H}^* \to \mathbb{A}$, that for any $\alpha, \beta \in \mathbb{R}$, $h_1, h_2, h_3 \in \mathbb{H}^*$ and $a \in \mathbb{A}$ satisfies the following conditions:

$i)$ $< h,h > \ \geq 0$,
$ii)$ $< h,h >= 0$ iff $h = 0$,
$iii)$ $< \alpha h_1 + \beta h_2, h_3 >= \alpha < h_1, h_3 > + \beta < h_2, h_3 >$
$iv)$ $< h_1, h_2 >=< h_2, h_1 >^*$
$v)$ $< ah_1, h_2 >= a < h_1, h_2 >$

A C^*-valued inner product is linear in its first argument and conjugate linear in its second argument. Having the $\mathbb{A}$-valued inner product, a norm as $\|h\| = \| < h,h > \|_{\mathbb{A}}^{1/2}$ can be defined on $\mathbb{H}^*$, where $\|.\|_{\mathbb{A}}$ is the norm on $\mathbb{A}$. If $\mathbb{H}^*$ is complete with this norm then the pair $(\mathbb{H}^*, < .,. >)$ is called a Hilbert $\mathbb{A}$-module. If there is no confusion we will show this space by $\mathbb{H}^*$.

Example 2.1. Any Hilbert space on $\mathbb{R}$ is a $\mathbb{A}$-module where $\mathbb{A} = \mathbb{R}$.

Example 2.2. Suppose $\mathbb{A}$ is a C^* algebra and let $\mathbb{H}^* = \mathbb{A}$. For a_1 and $a_2 \in \mathbb{A}$, define the inner product by $< a_1, a_2 >= a_1\ a_2^*$ then $(\mathbb{H}^*, < .,. >)$ is a $\mathbb{A}$-module.

Example 2.3. Let $\mathbb{H}^* = \mathbb{R}^n$ and $\mathbb{A}$ be the space of $n \times n$ matrices. For h_1 and $h_2 \in \mathbb{H}^*$ define $< h_1, h_2 >:= h_1\ h_2^*$. Then $(\mathbb{H}^*, < .,. >)$ is a $\mathbb{A}$-module and $\|h\| = \|h\ h^*\|_{\mathbb{A}}^{1/2}$.

Example 2.4. In general, suppose $\mathbb{H}$ is a Hilbert space with the inner product $< .,. >_{\mathbb{H}}$. Let $\mathbb{H}^* = \mathbb{H}^m$ and $\mathbb{A}$ the space of $m \times m$ matrices. For $h_1, h_2 \in \mathbb{H}^*$, define $< h_1, h_2 >= [< h_i, h_j >_{\mathbb{H}}]_{i,j=1}^m$ then $(\mathbb{H}^*, < .,. >)$ is a $\mathbb{A}$-module. We will denote this space by $\mathbb{H}^{*m}$.

The following proposition shows that the properties of a real valued inner product can be extended to an $\mathbb{A}$-valued inner product.

Proposition 2.1. *In a Hilbert $\mathbb{A}$-module space for any $h, h_1, h_2 \in \mathbb{H}^*$ and $a, b \in \mathbb{A}$, we have*

i) $\|ah\| \leq \|a\|_{\mathbb{A}} \|h\|$,
ii) $\| < h_1, h_2 > \|_{\mathbb{A}} \leq \|h_1\| \|h_2\|$,
iii) $< h_1, h_2 >< h_1, h_2 > \leq \|h_1\|^2 < h_1, h_2 >$.

The following lemma will be used to prove Proposition 4.1

Lemma 2.1. *Let*

(a) $\mathbb{H}^* = \mathbb{A}$, *where $\mathbb{A}$ is a C^* algebra (The Hilbert C^*- module in Example 2.2) or*
(b) $\mathbb{H}^* = \mathbb{H}^{*m}$, *where m is a natural number and $\mathbb{H}$ is a Hilbert space (the Hilbert C^*- module in Example 2.4),*

then the function $l : \mathbb{H}^* \to \mathbb{R}$ is in $L(\mathbb{H}^*, R)$ iff for some $h_* \in \mathbb{H}^*$ and $\phi \in L(\mathbb{A}, \mathbb{R})$, $l(\cdot) = \phi(< \cdot, h_* >)$.

Proof. It is trivial that if $h_* \in \mathbb{H}^*$ and $\phi \in L(\mathbb{A}, \mathbb{R})$, and if we let $l(\cdot) = \phi(< \cdot, h_* >)$ then $l \in L(\mathbb{H}^*, \mathbb{R})$. To prove the reverse, suppose $l \neq 0$, otherwise we could take $h_* = 0$ and $\phi = 0$. In case (a) we can simply take h_* to be the identity element in $\mathbb{A}$ and $\phi = l$. In case (b) it can easily be shown that any linear functional l, for some $h_* = [h_{*1}, \cdots, h_{*m}]'$ can be written as $l(h) = l([h_1, \cdots, h_m]') = \sum_{i=1}^{m} < h_i, h_{*i} >_{\mathbb{H}}$ where $l(h_*) \neq 0$, and $< ., . >_{\mathbb{H}}$ is the inner product for the Hilbert space $\mathbb{H}$. Now consider $\phi =$ the trace as the linear functional from $\mathbb{A}$ to $\mathbb{R}$ then $l(h) = \phi(< h, h_* >)$. □

In Section 4, we will give the definition of C^*-module random variables and study their characteristics. To do so we need to know some properties of Banach-valued random variables. Thus, in the next section, we have a brief review on these random variables.

3. Banach valued random variables

Let $\mathbb{B}$ be a separable Banach space and denote the collection of its Borel sets by $\mathscr{B}$. Suppose $(\Omega, \mathscr{F}, P)$ is a probability space. A measurable function $X : \Omega \to \mathbb{B}$ is called a Banach valued random variable. For simplicity, we will call a Banach valued random variable a $\mathbb{B}$-valued random variable. Most of the properties of real valued random variables can be extended to $\mathbb{B}$-valued random variables. Some of these properties are as follows:

(a) The collection of all $\mathbb{B}$-valued random variable is closed under formation of pointwise limits.
(b) If X is a $\mathbb{B}$-valued random variable then there exists a sequence $\{X_n\}$ of simple $\mathbb{B}$- valued random variables such that $X = \lim_n X_n$ and $\|X_n\| \leq \|X\|$ for all n.
(c) The collection of all $\mathbb{B}$-valued random variables is a vector space.

Proof: see Ledoux and Talagrand [7].

For $\mathbb{B}$-valued random variables characteristic functional, mean and covariance operator are defined as follows.

Definition 3.1. Characteristic functional of a $\mathbb{B}$-valued r.v. X, $\varphi(l)$, for $l \in L(\mathbb{B}, \mathbb{R})$ is defined as:

$$\varphi(l) = Ee^{i\, l(X)}.$$

The functional φ is a continuous positive definite functional on $L(\mathbb{B}, \mathbb{R})$ and $|\varphi(l)| \leq 1$.

Definition 3.2. The mean of a $\mathbb{B}$-valued random variable X is the element μ of $\mathbb{B}$ that for any $l \in L(\mathbb{B},\mathbb{R})$ satisfies:

$$l(\mu) = El(X).$$

The mean of X in general, does not exist, but if $E||X|| < \infty$ then μ exists and $||\mu|| \leq E||X||$.

Remark 3.1. The set of $\mathbb{B}$-valued random variables with finite mean is a Banach space. In other words the mean operator is a linear operator on this set. See Ledoux and Talagrand [7, page 36].

Remark 3.2. Let Y be a $\mathbb{B}$-valued random variable and X be a random object, where both are defined on a same probability space. Then we can define conditional expectation of Y given X, $E(Y|X)$, in a usual manner and under very general conditions $EY = EE(Y|X)$.

Definition 3.3. The covariance operator of a $\mathbb{B}$-valued random variable X with the mean of μ, is a bilinear operator defined on $L(\mathbb{B},\mathbb{R}) \times L(\mathbb{B},\mathbb{R})$ as:

$$S(l_1,l_2) = El_1(X-\mu)l_2(X-\mu).$$

The covariance operator is a positive definite symmetric bilinear operator. Of course this operator necessarily does not exist but if $E||X||^2 < \infty$, this operator exists.

Now using the mean and the covariance operator we can define a Gaussian distribution for $\mathbb{B}$-valued random variables.

Definition 3.4. A $\mathbb{B}$-valued random variable X is said to have a Gaussian or normal distribution if for any $l \in L(\mathbb{B},\mathbb{R})$ the random variable $l(X)$ has a univariate Gaussian distribution. Or equivalently, if and only if the characteristic functional of X, $\varphi(l)$ has the form of

$$\varphi(l) = exp[i\, l(\mu) - \frac{1}{2}S(l,l)].$$

where μ and S are, respectively the mean and covariance operator of X.

If $\mathbb{B}$ is a Hilbert space, denoted by $\mathbb{H}$, with the inner product $< .,. >_{\mathbb{H}}$, we call a $\mathbb{B}$-valued random variable $\mathbb{H}$-valued random variable. In this case the above definitions reduce to the following.

Definition 3.5. The characteristic functional of a $\mathbb{H}$-valued random variable X, $\varphi(h)$, for any $h \in \mathbb{H}$ is defined as:

$$\varphi(h) = Ee^{i<X,h>}.$$

Definition 3.6. The mean of a $\mathbb{H}$-valued random variable X is the element μ in $\mathbb{H}$ which for any $h \in \mathbb{H}$ satisfies:

$$< h,\mu >= E < X,h > .$$

Definition 3.7. The covariance operator of a $\mathbb{H}$-valued random variable X with the mean of μ, is a bilinear operator on $\mathbb{H} \times \mathbb{H}$ defined as:

$$< Sh_1,h_2 >= E < (X-\mu),h_1 >< (X-\mu),h_2 > .$$

Definition 3.8. A $\mathbb{H}$-valued random variable X has a Gaussian distribution if its characteristic functional, $\varphi(h)$, has the form of

$$\varphi(h) = exp[i < \mu, h > -\frac{1}{2} < Sh, h >].$$

where μ and S are the mean and covariance operator of X, respectively.

The central limit theorem is one of the oldest results in probability theory. It is still a main part of classical probability theory, and is basic to asymptotic statistical theory. Now, we review this theorem for independent identically distributed(iid) $\mathbb{B}-$valued random variables.

Let $X_1, \cdots, X_n$ be a random sample from a $\mathbb{B}$-valued random variable X with the distribution m and mean of μ. Let $Z_n = \sqrt{n}(\frac{1}{n}\sum X_i - \mu)$ and m_n be the distribution of Z_n. It is said that the random variable X satisfy the central limit theorem (CLT) in $\mathbb{B}$ if there exists a $\mathbb{B}$ valued random variable with a Gaussian distribution γ such that $\mu_n \to \gamma$ weakly (see Billingsley [2]). It is well known that on the real line a random variable X satisfies the CLT if and only if $EX^2 < \infty$ and if X satisfies the CLT, the sequence Z_n converges weakly to a normal distribution with mean zero and variance EX^2 (see Feller [3]). The sufficiency of the condition $E\|X\|^2 < \infty$ for a random variable X to satisfy the CLT extends to the case where $\mathbb{B}$ is of finite dimension. For a recent review of the CLT in the setting of Banach spaces see Ledoux and Zinn [8]. In general, very bad situations can occur and strong assumption on the distribution of X do not guarantee that X satisfies the CLT. For an example of almost surely bounded $\mathbb{B}$-valued X that does not satisfies the CLT see Araujo and Gine [1]. But we have some positive results for spaces of type 2.

Definition 3.9. A separable Banach space $\mathbb{B}$ is called of type p, $1 \le p \le 2$, if there exists a constant C such that if $\{X_i\}_{i=1}^n$ is any finite set of independent $\mathbb{B}$-valued random variables with $E\|X_i\|^p < \infty$, then

$$E\|\sum_{i=1}^n X_i\|^p \le C\sum_{i=1}^n E\|X_i\|^p.$$

The following theorem extends to type 2 spaces the sufficient condition on the line for the CLT. For the proof see Ledoux and Talagrand [7].

Theorem 3.1. *Let X be a $\mathbb{B}$-valued random variable with $E\|X\|^2 < \infty$ with values in a separable Banach space of type* 2. *Then X satisfies the CLT.*

Remark 3.3. It can be shown that any Hilbert space is of type 2.

Remark 3.4. Suppose $\mathbb{B}$ is a linear space and it becomes a Banach space with two norms $\|.\|_1$ and $\|.\|_2$. If two norms are equivalent then the Banach spaces of $(\mathbb{B}, \|.\|_1)$ and $(\mathbb{B}, \|.\|_2)$ have the same type.

4. $\mathbb{H}^*$-valued random variables

The Hilbert C^*-module spaces are Banach spaces so definitions and results in the previous section are valid for random variables which take their values from a Hilbert C^*-module. As Hilbert C^*-module spaces carry an inner product, it would be interesting to know what role this inner product can play in the characteristics of a $\mathbb{H}^*$-valued random variable.

4.1. *Characteristics of* $\mathbb{H}^*$*-valued random variables*

A $\mathbb{H}^*$-valued random variable is a $\mathbb{B}$-valued random variable whose corresponding Banach space is a Hilbert C^*-module. The following propositions show that three possible definitions for the mean, covariance operator and Gaussian distribution for $\mathbb{H}^*$-valued random variables are equivalent.

Proposition 4.1. *Under the conditions of Lemma 2.1, let X be a $\mathbb{H}^*$-valued random variable such that $\mathrm{E}\|X\| < \infty$, then its mean is the element μ in $\mathbb{H}^*$ that satisfies one of the following equivalent conditions.*

i) For any $h \in \mathbb{H}^*$ and $\phi \in L(\mathbb{A},\mathbb{R})$

$$\phi(< \mu, h >) = \mathrm{E}\phi(< X, h >). \tag{4.1}$$

ii) For any $k \in L(\mathbb{H}^*,\mathbb{A})$ and $\phi \in L(\mathbb{A},\mathbb{R})$

$$\phi(k(\mu)) = \mathrm{E}\phi((k(X))). \tag{4.2}$$

iii) For any $l \in L(\mathbb{H}^*,\mathbb{R})$

$$l(\mu) = \mathrm{E}l(X). \tag{4.3}$$

Proof. To prove the assertion we show *iii* $\Rightarrow$ *ii* $\Rightarrow$ *i* $\Rightarrow$ *iii*. *iii* is the definition for the mean of X as a $\mathbb{B}$-valued random variable. Since for any $k \in L(\mathbb{H}^*,\mathbb{A})$ and $\phi \in L(\mathbb{A},\mathbb{R})$, ϕk is an element of $L(\mathbb{H}^*,\mathbb{R})$ then *iii* $\Rightarrow$ *ii*. As for any $h \in \mathbb{H}^*$ and $\phi \in L(\mathbb{A},\mathbb{R})$,$\phi(< ., h >)$ is an element of $L(\mathbb{H}^*,\mathbb{R})$ we have *ii* $\Rightarrow$ *i*. This *i* $\Rightarrow$ *iii* is a result of Lemma 2.1. □

Proposition 4.2. *Under the conditions of Lemma 2.1, if X is a $\mathbb{H}^*$-valued random variable with the mean of μ and $\mathrm{E}\|X\|^2 < \infty$, then its covariance operator S is a bilinear operator that satisfies one of the following equivalent conditions.*

i) For any $h_1, h_2 \in \mathbb{H}^*$ and $\phi_1, \phi_2 \in L(\mathbb{A},\mathbb{R})$

$$S((\phi_1, h_1),(\phi_2, h_2)) = \mathrm{E}\phi_1(< X-\mu, h_1-\mu >)\phi_2(< X-\mu, h_2-\mu >). \tag{4.4}$$

ii) For any $k_1, k_2 \in L(\mathbb{H}^*,\mathbb{A})$ and $\phi_1, \phi_2 \in L(\mathbb{A},\mathbb{R})$

$$S(\phi_1 k_1, \phi_2 k_2) = E\phi_1 k_1(X-\mu)\phi_2 k_2(X-\mu,). \tag{4.5}$$

iii) For any $l_1, l_2 \in L(\mathbb{H}^*,\mathbb{R})$

$$S(l_1, l_2) = \mathrm{E}l_1(X-\mu)l_2(X-\mu). \tag{4.6}$$

Proof. The proof is similar to the previous proposition.

Proposition 4.3. *Under the conditions of Lemma 2.1, a $\mathbb{H}^*$-valued random variable X has a Gaussian distribution iff one of the following equivalent conditions is true.*

i) For any $h \in \mathbb{H}^*$ and $\phi \in L(\mathbb{A},\mathbb{R})$, $\phi(< X, h >)$ has a univariate normal distribution.
ii) For any $k \in L(\mathbb{H}^*,\mathbb{A})$ and $\phi \in L(\mathbb{A},\mathbb{R})$, $\phi((k(X))$ as a univariate normal distribution.
iii) For any $l \in L(\mathbb{H}^*,\mathbb{R})$, $l(X)$as a univariate normal distribution.

Proof. The proof is similar to Proposition 4.1.

4.2. *Covariance of two $\mathbb{H}^*$-valued random variables*

One of the characteristics of the joint distribution of two real valued random variables is their covariance. This characteristic, cannot be easily extended for random variables that take their values in a space without a product operation. In Grenander [4] second moment operator between two $\mathbb{H}$-valued random variables is extended as follows.

Definition 4.1. The second moment operator between two $\mathbb{H}$-valued random variables X and Y, denoted by Δ, is an operator on $\mathbb{H}$ that for $h \in \mathbb{H}$ is defined as:

$$\Delta(h) = E < X, h >_{\mathbb{H}} Y.$$

Now using this idea and the inner product in $\mathbb{H}^*$ we extend the definition and properties of the covariance for two $\mathbb{H}$-valued random variables.

Definition 4.2. Let X and Y be two $\mathbb{H}^*$-valued random variables defined on the same probability space. We define the covariance of X and Y, denoted by $Cov(X,Y)$ as :

$$Cov(X,Y) = E < X - \mu_X, Y - \mu_Y >, \tag{4.7}$$

where μ_X and μ_Y are the mean of X and Y, respectively. Note that if $E\|X\|^2$ and $E\|Y\|^2$ are finite then $Cov(X,Y)$ exists and also $Cov(Y,X) = Cov(X,Y)^*$.

Remark 4.1. $Cov(X,Y) = E < X - \mu_X, Y - \mu_Y >= E < X,Y > - < \mu_X, \mu_Y >$.

We can define the variance of a $\mathbb{H}^*$-valued random variable X as $Var(X) = Cov(X,X) = E < X - \mu, X - \mu >$. Note that $Var(X)$ is a nonnegative element of $\mathbb{A}$ and if the variances of X and Y are invertible, we can define the correlation between X and Y as $\rho(X,Y) = Var(X)^{-1/2} Cov(X,Y) Var(Y)^{-1/2}$.

Suppose Y is a $\mathbb{H}^*$-valued random variable and X is a random object and X and Y are defined on the same probability space. If we define the conditional variance of Y given X, as

$$Var(Y|X) = E(< Y,Y > |X) - < E(Y|X), E(Y|X) >,$$

we have the following result, which can be used in generalizing the Rao-Blackwell theorem for $\mathbb{H}^*$-valued random variables.

Proposition 4.4. *Under the above conditions we have*

$$Var(Y) = E[Var(Y|X)] + Var[E(Y|X)].$$

Proof. The proof is similar to the real valued case. □

The following propositions show that the properties of covariance between real valued random variables can be extended to the covariance of $\mathbb{H}^*$-valued random variables.

Proposition 4.5. *Let X and Y be two independent $\mathbb{H}^*$-valued random variables then we have $Cov(X,Y) = 0$.*

Proof. Without loss of generality we assume $\mu_Y = \mu_X = 0$. Let $Z =< X,Y >$ then as an $\mathbb{A}$-valued random variable $E(Z|Y) = E(< X,Y > |Y)$ and from (4.1) for any $\phi \in L(\mathbb{A},\mathbb{R})$ we have

$$\phi(< E(X|Y),Y >) = E(\phi(< X,Y >)|Y).$$

Since X and Y are independent, $E(X|Y) = E(X) = 0$ thus $E(Z|Y) =< 0,Y >$. A further application of (4.1) shows that $EZ = E < X,Y >= 0$. □

Proposition 4.6. *Let $X_1,\cdots,X_n$ and $Y_1,\cdots,Y_m$ be $\mathbb{H}^*$-valued random variables defined on the same probability space, then*

i) For any $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^m \subset \mathbb{R}$,

$$Cov(\sum_{i=1}^n a_iX_i,\sum_{i=1}^m b_iY_i) = \sum_{i=1}^n\sum_{j=1}^m a_ib_jCov(X_i,Y_j)$$

ii) For any $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^m \subset \mathbb{A}$,

$$Cov(\sum_{i=1}^n A_iX_i,\sum_{i=1}^m B_iY_i) = \sum_{i=1}^n\sum_{j=1}^m A_iCov(X_i,Y_j)B_j^*$$

Proof. Clear. □

We have the following result for the independent case.

Corollary 4.1. *If $X_1,\cdots,X_n$ are independent $\mathbb{H}^*$-valued random variables then*

(i) For any $\{a_i\}_{i=1}^n \subset \mathbb{R}$,

$$Var(\sum_{i=1}^n a_iX_i) = \sum_{i=1}^n a_i^2Var(X_i)$$

ii) For any $\{A_i\}_{i=1}^n \subset \mathbb{A}$,

$$Var(\sum_{i=1}^n A_iX_i) = \sum_{i=1}^n A_iVar(X_i)A_i^*$$

4.3. *The CLT for $\mathbb{H}^{*m}$-valued random variables*

Now we turn to the CLT for the $\mathbb{H}^{*m}$-valued random variables where $\mathbb{H}^{*m}$ is defined in Example 2.4.

We will show that the finiteness of the second moment of the norm is sufficient for a $\mathbb{H}^{*m}$-valued random variable to satisfy the CLT.

Proposition 4.7. *Assume X is a $\mathbb{H}^{*m}$-valued random variable and* $\mathrm{E}\|X\|^2 < \infty$. *Then X satisfies the CLT.*

Proof. By Theorem 3.1 it is enough to show that $\mathbb{H}^{*m}$ spaces are of type 2. As a Hilbert space $\mathbb{H}^{*m}$ is of type 2 according to Remarks 4 and 5 it suffices to show that two norms $\|x\|_2 = \| < x,x > \|_{\mathbb{A}}^{1/2}$ and $\|x\|_1 = \{\sum_{i=1}^m < x_i,x_i >_{\mathbb{H}}\}^{1/2}$ are equivalent on $\mathbb{H}^{*m}$. To do this, we show for any

$x \in \mathbb{H}^*{}_m$, $\|x\|_1 \leq \sqrt{m}\|x\|_2$. Let $B = [< x_i, x_j >_{\mathbb{H}}]_{i,j=1}^m$ and for $i = 1, \cdots, m$, λ_i be the eigenvalues of B in nondecreasing order. We know $\|x\|_1^2 = \sum_{i=1}^m \lambda_i$ and since

$$\begin{aligned} \|x\|_2^2 &= \| < x, x > \|_{\mathbb{A}} \\ &= \sup\left\{\sqrt{a^* B\, B^* a};\ a \in \mathbb{R}^m, a^* a = 1\right\} \\ &= \sqrt{\sup\{a^* B\, B^* a;\ a \in \mathbb{R}^m, a^* a = 1\}} \\ &= \lambda_m \\ &\geq \lambda_i \quad i = 1, \cdots, m, \end{aligned}$$

we have $\|x\|_1 \leq \sqrt{m}\|x\|_2$. □

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References

[1] Araujo, A. and Gine, E. (1980). *The central limit theorem for real and Banach valued random variables*. Wiley series in Probability and Mathematical Statistics.
[2] Billingsley, P. (1968). *Convergence of probability measures*. Wiley, New York.
[3] Feller, W. (1971). *An introduction to probability theory and its applications*, volume I. Wiley, New York.
[4] Grenander, U. (1963). *Probabilities on Algebraic Structures*. Almqvist & Wiksell, Stockholm.
[5] Kaplansky, I. (1953). Modules over operator algebras. *Amer. J. Math.*, 75:839–858.
[6] Lance, E. C. (1995). *Hilbert C^* modules*. Number 210 in (LMS) Lecture Notes, Series. Cambridge University Press.
[7] Ledoux, M. and Talagrand, M. (1991). *Probability in Banach Spaces*. Springer-Verlag, Heidelberg.
[8] Ledoux, M. and Zinn, J. (2003). *Probabilistic Limit Theorems in the Setting of Banach Spaces*, volume 2 of *Handbook of the Geometry of Banach Spaces*, chapter 27, pages 1177–1200. North-Holland Publishing Co.
[9] Packer, J. A. and Rieffel, M. A. (2003). Wavelet filter functions, the matrix completion problem, and projective modules over $c(\mathbf{T^n})$. *J. Fourier Anal. Appl.*, 9(2):101–116.
[10] Packer, J. A. and Rieffel, M. A. (2004). Projective multi-resolution analyses for $l^2(r^2)$. *J. Fourier Anal. Appl.*, 10:439–464.
[11] Paschke, W. L. (1973). Inner product modules over b^*-algebras. *Trans. Amer. Math. Soc.*, 182:443–464.
[12] Popovich, D. (1998). Orthogonal decomposition in Hilbert c^*- modules and stationary processes. *Acta Math. Univ. Comenianae*, LXVII(2):217–230.
[13] Ramsay, J. O. and Silverman, B. W. (1997). *Applied Functional Data Analysis*. Springer Series in Statistics. Springer, New York.