Journal of Statistical Theory and Applications, Vol. 14, No. 4 (December 2015), 383-398

Likelihood-based inference for the power half-normal distribution

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Received 14 January 2015

Accepted 27 May 2015

Abstract

In this paper we consider an extension of the half-normal distribution based on the distribution of the maximum of a random sample. It is shown that this distribution belongs to the family of beta generalized half-normal distributions. Properties of its density are investigated, maximum likelihood estimation is discussed and the Fisher information matrix is derived. A real data illustration is presented, and comparisons with alternative extensions of the half-normal distribution reveal good performance of the proposed model.

Keywords: Distribution of the maximum of a sample; Half-normal distribution; Maximum likelihood.

2000 Mathematics Subject Classification: 22E46, 53C35, 57S20

1. Introduction

Lehmann (1953) proposes an asymmetric family of distributions with distribution function (d.f.)

$$\mathscr{F}_F(z;\alpha) = \{F(z)\}^{\alpha}, \quad z \in \mathbb{R} \tag{1.1}$$

where F is a distribution function and α is a rational number. Clearly, when α is an integer, the distribution of the maximum of a sample follows. Durrans (1992) extended the definition of (1.1) by allowing $\alpha \in \mathbb{R}^+$, referring to the resulting distributions as the *fractional order statistics*. Assuming F to be absolutely continuous with density $f = dF$ in (1.1), the density of a random variable, Z, from such a distribution is given by

$$\varphi_F(z;\alpha) = \alpha f(z)\{F(z)\}^{\alpha-1}, \qquad \alpha \in \mathbb{R}^+. \tag{1.2}$$

We use the notation $Z \sim FO_F(\alpha)$.

Remark 1. It follows that function (1.2) is a density function provided $z \in \mathbb{R}$ or $z \in \mathbb{R}^+$, from which alternative distributions can be defined with support $\mathbb{R}$ or $\mathbb{R}^+$.

Durrans (1992) considered the case of (1.2) in which F is the d.f. of the standard normal distribution, Φ, referring to the resulting distribution, $FO_\Phi(\alpha)$, with density

$$\varphi_\Phi(z;\alpha) = \alpha\phi(z)\{\Phi(z)\}^{\alpha-1}, \quad z \in \mathbb{R}, \;\; \alpha \in \mathbb{R}^+, \tag{1.3}$$

as being the *generalized Gaussian* distribution. Gupta and Gupta (2008) also considered the $FO_\Phi(\alpha)$ distribution in some detail. They referred to the class of distributions with density (1.3) as the *power-normal model*.

Lehmann (1953), Durrans (1992) and Gupta and Gupta (2008) can be consulted for the fundamental properties of the $FO_\Phi(\alpha)$ distribution. Pewsey et al. (2012) show that the Fisher information matrix for the location-scale extension of the power-normal model is nonsingular for $\alpha = 1$, that is, the ordinary location-scale normal model.

In Durrans (1992), the generalized Gaussian distribution is also a special case of the so-called Beta-normal distribution of Eugene et al. (2002). Jones (2004) considered an extension of the construction in Eugene et al. (2002) which includes any distribution function and not just the standard normal distribution.

On the other hand, the generalized half-normal distribution is introduced in Cooray and Ananda (2008) as an alternative to the Gamma, Weibull, log-normal and Birnbaum-Saunders distributions to model positive life time data. We say that X is a random variable with generalized half-normal (GHN) distribution with scale parameter σ and shape parameter α, if its density function is given by

$$h(x;\sigma,\alpha) = \sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{x}\right)\left(\frac{x}{\sigma}\right)^{\alpha} exp\left[-\frac{1}{2}\left(\frac{x}{\sigma}\right)^{2\alpha}\right], \; x > 0, \sigma > 0, \alpha > 0.$$

Denoting $X \sim GHN(\sigma,\alpha)$, note that $GHN(\sigma,\alpha = 1) \equiv HN(\sigma)$, that is, one obtains the half-normal model with scale parameter σ. This family is extended in Pescim et al. (2010), where a four parameter family is generated.

This paper focuses on studying the distribution that is generated when we consider the density (distribution) function in (1.2) as the density (distribution) function of the half-normal distribution. The half-normal distribution introduced is called the power half-normal (PHN) distribution. The PHN family is a subfamily of the distributions considered in Pescim et al. (2010). The distribution PHN has only two parameters and can be used for fitting positive data from reliability or survival experiments being thus an alternative to half-normal, gamma and Weibull distributions, among others.

The paper is organized as follows. In Sec. 2 we present the power-half-normal distribution. Basic properties such as quantiles, risk functions, characterizations and moments derivation are discussed. Sec. 3 deals with inferential aspects such as likelihood function, likelihood equations and Fisher information matrix. In Sec. 4, we illustrate the importance of the new distribution by applying it to three real data sets using maximum likelihood estimation. Results reveal that the proposed PHN model can outperform competing alternatives previously proposed.

2. Power half-normal distribution

Definition 1. A random variable Z is said to follow a power half-normal distribution with scale parameter σ and shape parameter α if its probability density function (pdf) is given by:

$$f_Z(z;\sigma,\alpha) = \frac{2\alpha}{\sigma}\phi\left(\frac{z}{\sigma}\right)\left(2\Phi\left(\frac{z}{\sigma}\right)-1\right)^{\alpha-1},$$

where $\sigma > 0, \alpha > 0, z > 0$ and $\phi(\cdot)$ $(\Phi(\cdot))$ denotes the density (distribution) function for the standard normal density (distribution) function. We use the notation $Z \sim PHN(\sigma,\alpha)$.

Definition 2. The distribution function for $Z \sim PHN(\sigma,\alpha)$ is given by

$$F_Z(z;\sigma,\alpha) = \left(2\Phi\left(\frac{z}{\sigma}\right)-1\right)^{\alpha},$$

where $\sigma > 0, \alpha > 0, z > 0$.

Fig. 1 and 2 depicts graphically the density function for model PHN for some values of the parameters σ and α, whereas Figure 3 shows plots of the distribution function for some values of parameters σ and α.

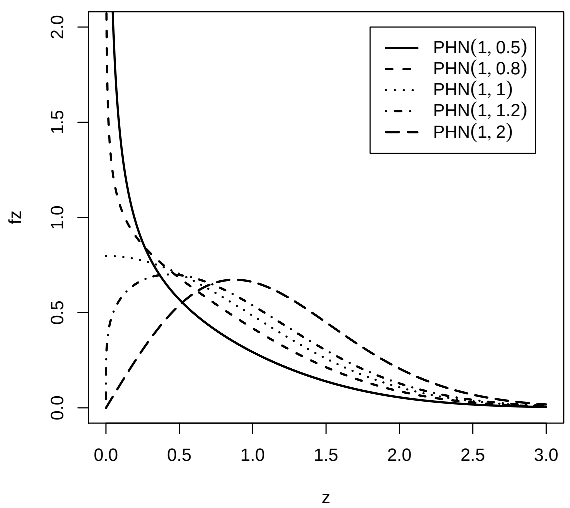

Fig. 1. Probability density functions of Z for $\sigma = 1$ and $\alpha = 0.5, 0.8, 1, 1.2, 2$.

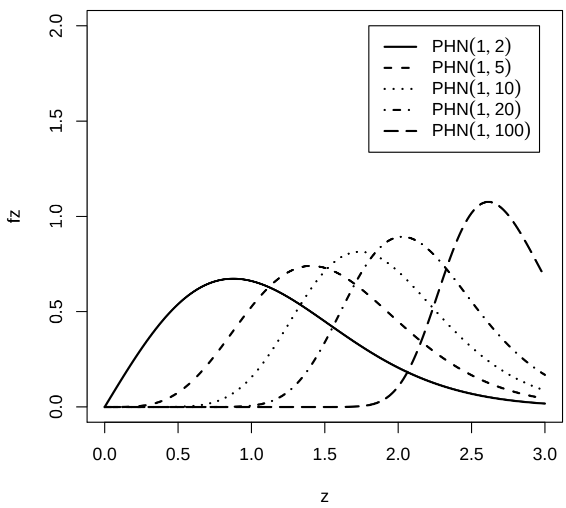

Fig. 2. Probability density functions of Z for $\sigma = 1$ and $\alpha = 2, 5, 10, 20, 100$.

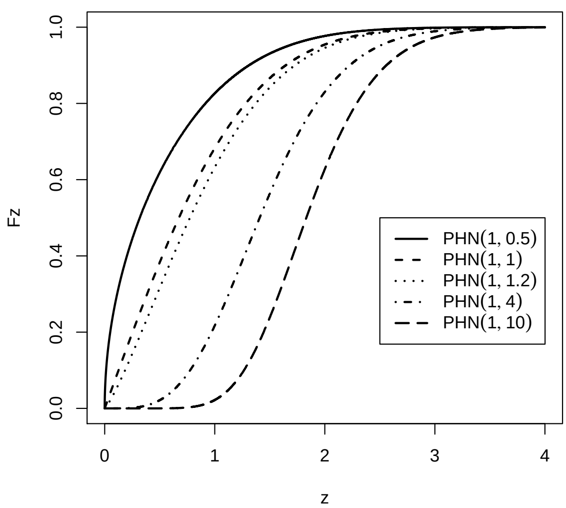

Fig. 3. Distribution functions of Z for for $\sigma = 1$ and $\alpha = 0.5, 1, 1.2, 4, 10$.

2.1. *Main properties*

In the following, quantiles, hazard and survival functions are derived for model PHN. Thus, letting $Z \sim PHN(\sigma, \alpha)$, we have

$$Q(p) = \sigma\Phi^{-1}\left(\frac{1+p^{1/\alpha}}{2}\right), \quad 0 < p < 1,$$

and

1. $first\ quartile = \sigma\Phi^{-1}\left(\frac{1+4^{1/\alpha}}{2^{2/\alpha+1}}\right)$
2. $Median(Z) = \sigma\Phi^{-1}\left(\frac{1+2^{1/\alpha}}{2^{1/\alpha+1}}\right)$
3. $third\ quartile = \sigma\Phi^{-1}\left(\frac{3+4^{1/\alpha}}{2^{2/\alpha+1}}\right)$

The corresponding survival and the hazard rate functions are, respectively, given by

$$S(z) = 1 - F_Z(z) = 1 - \left(2\Phi\left(\frac{z}{\sigma}\right) - 1\right)^{\alpha},$$

and

$$h(z) = \frac{f(z)}{S(z)} = \frac{\frac{2\alpha}{\sigma}\phi\left(\frac{z}{\sigma}\right)\left(2\Phi\left(\frac{z}{\sigma}\right)-1\right)^{\alpha-1}}{1-\left(2\Phi\left(\frac{z}{\sigma}\right)-1\right)^{\alpha}}.$$

Fig. 4 and 5 depicts plots for the survival and hazard rate functions for the half power-normal distribution. Notice that there are instances that the hazard function is initially decreasing and then increasing being thus of the bathtub shape.

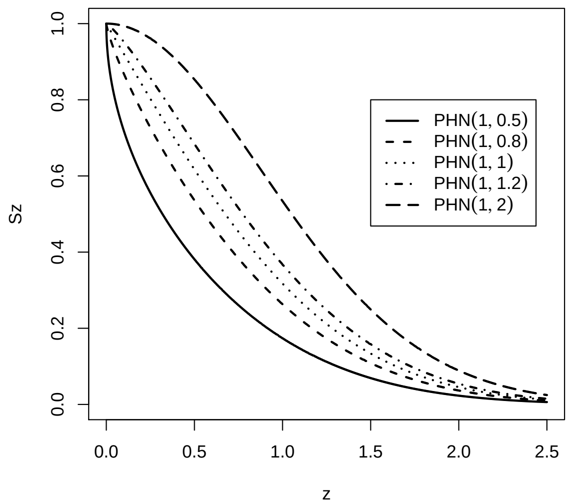

Fig. 4. Survival function of Z for $\sigma = 1$ and $\alpha = 0.5, 0.8, 1, 1.2, 2$.

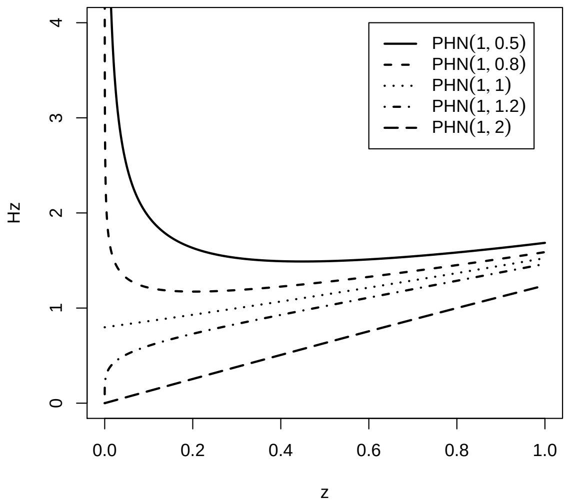

Fig. 5. Hazard rate function of Z for $\sigma = 1$ and $\alpha = 0.5, 0.8, 1, 1.2, 2$.

Note that the functions $F_Z(z;\sigma,\alpha)$, $Q(p;\sigma,\alpha)$, $s(z;\sigma,\alpha)$ and $h(z;\sigma,\alpha)$ only depend on $\Phi(\cdot)$. So, they can be easily calculated.

2.2. *Characterizations of PHN distribution*

In designing a stochastic model for a particular modeling problem, investigators will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterizations results have been reported in the literature. These characterizations have been established in many different directions. In this Section, we present yet another characterizations of PHN distribution. These characterizations are based on conditional expectations of a function of the random variable.

Here, we employ a single function ψ of X and state characterization results in terms of $\psi(X)$.

Proposition 1. *Let $X : \Omega \to (a,b)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on (a,b) with $\lim_{x\to b^-} \psi(x) = 1$. Then for $\delta \neq 1$,*

$$E[\psi(X) \mid X \leq x] = \delta\psi(x), \quad x \in (a,b), \tag{2.1}$$

if and only if

$$\psi(x) = (F(x))^{\frac{1}{\delta}-1}, \quad x \in (a,b)$$

Proof. Is straightforward. □

A more general form of Eq. (2.1) is given below.

Proposition 2. *Let* $X : \Omega \rightarrow (a,b)$ *be a continuous random variable with* cdf F. *Let* $\psi(x)$ *be a differentiable function on* (a,b) *with* $\lim_{x\rightarrow a^+} \psi(x) = \delta > 1$ *and* $\lim_{x\rightarrow b^-} \psi(x) = \infty$. *Then*

$$E\left[(\psi(X))^{\delta} \mid X \leq x \right] = \delta\,(\psi(x))^{\delta-1}, \quad x \in (a,b), \tag{2.2}$$

if and only if

$$\psi(x) = \delta\left[1 + (F(x))^{\frac{1}{\delta-1}}\right]^{-1}, \quad x \in (a,b). \tag{2.3}$$

Proof. From Eq. (2.2), we have

$$\int_a^x (\psi(u))^{\delta} f(u)\,du = \delta\,(\psi(x))^{\delta-1} F(x).$$

Taking derivatives from both sides of the above equation and rearranging terms, we arrive at

$$\frac{f(x)}{F(x)} = (\delta - 1)\left\{ -\frac{\psi'(x)}{\psi(x)} + \frac{\psi'(x)}{\psi(x) - \delta} \right\}.$$

Integrating both sides of this equation from x to b and using the condition $\lim_{x\rightarrow b^-} \psi(x) = \infty$, we obtain Eq. (2.3).□

Remark 2. (a) Taking, e.g., $(a,b) = (0,\infty)$ and $\psi(x) = \left(2\Phi\left(\frac{x}{\sigma}\right) - 1\right)^{\frac{\alpha(1-\delta)}{\delta}}$, Proposition 1 gives a characterization of PHN distribution. (b) Taking, e.g., $(a,b) = (0,\infty)$ and $\psi(x) = \frac{\delta}{1+\left(2\Phi\left(\frac{x}{\sigma}\right)-1\right)^{\frac{\alpha}{\delta-1}}}$, Proposition 2 gives a characterization of PHN distribution.

2.3. *Moments*

Moments of the PHN model can be computed numerically using routine "integrate" from software R. The following proposition presents r-th moments of a random variable following the PHN distribution.

Proposition 3. *The r-th moment of the random variable* $Z \sim PHN(\sigma,\alpha)$*, is given by*

$$\mu_r = E(Z^r) = \alpha\sigma^r \kappa_r(\alpha), \quad r = 1, 2, ...$$

where $\kappa_r = \kappa_r(\alpha) = \int_0^1 \left(\Phi^{-1}\left(\frac{1+u}{2}\right)\right)^r u^{\alpha-1} du$*, are computed numerically.*

Proof. The definition of moment implies

$$\mu_r = E(Z^r) = \int_0^\infty \frac{2\alpha}{\sigma} z^r \phi\left(\frac{z}{\sigma}\right)\left(2\Phi\left(\frac{z}{\sigma}\right)-1\right)^{\alpha-1} dz.$$

The result follows after making the variable change $u = 2\Phi\left(\frac{z}{\sigma}\right) - 1$. □
Therefore, the first four moments are given by

1. $\mu_1 = E(Z) = \alpha\sigma\kappa_1$.
2. $\mu_2 = E(Z^2) = \alpha\sigma^2\kappa_2$.
3. $\mu_3 = E(Z^3) = \alpha\sigma^3\kappa_3$.
4. $\mu_4 = E(Z^4) = \alpha\sigma^4\kappa_4$.

Corollary 1. *The asymmetry and kurtosis coefficients are given, respectively, by*

$$\sqrt{\beta_1} = \frac{\kappa_3 - 3\alpha\kappa_1\kappa_2 + 2\alpha^2\kappa_1^3}{\sqrt{\alpha}(\kappa_2 - \alpha\kappa_1^2)^{3/2}},$$

$$\beta_2 = \frac{\kappa_4 - 4\alpha\kappa_1\kappa_3 + 6\alpha^2\kappa_1^2\kappa_2 - 3\alpha^3\kappa_1^4}{\alpha(\kappa_2 - \alpha\kappa_1^2)^2}.$$

Fig. 6 and 7 depict the skewness and kurtosis coefficients for some values of the parameter α.

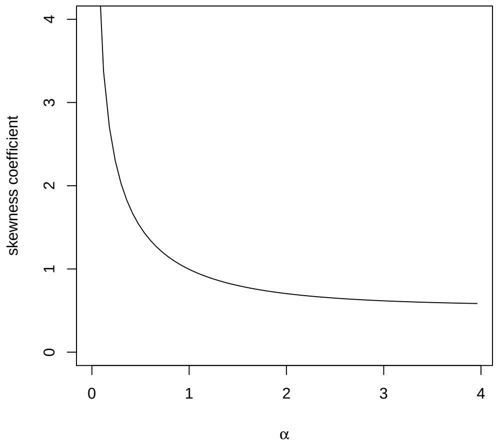

Fig. 6. Skewness coefficient for some values of α

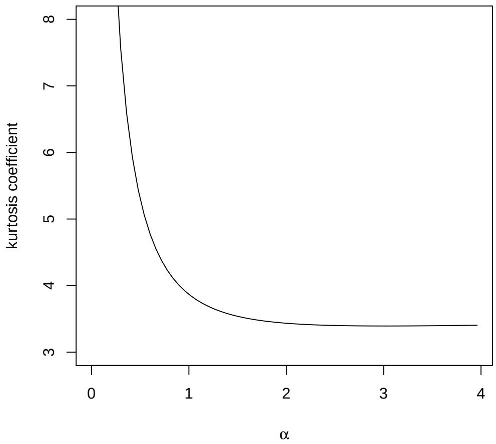

Fig. 7. Kurtosis coefficient for some values of α

2.4. *Shannon entropy*

The entropy of a random variable Z is a measure of its uncertainty
Shannon entropy measure is defined by

$$\mathbf{J}_S = -E(\log(f_Z(z))).$$

If follows, after extensive algebraic manipulations that Shannon entropy for the PHN model is:

$$\mathbf{J}_S = -\log(2) - \log(\alpha) + \log(\sigma) + \log(\sqrt{2\pi}) + \frac{\alpha\kappa_2}{2} + \frac{\alpha-1}{\alpha},$$

with κ_2 as given above. Notice that as $\alpha = 1$ (half-normal distribution), $\frac{\alpha\kappa_2}{2} = \frac{E(Z^2)}{2\sigma^2}$ and $E(Z^2) = \sigma^2$, where $E(Z^2)$ is the second moment of the half-normal distribution. Then one obtains, as a particular case, the Shannon entropy for the half-normal distribution (see Ahsanullah et al. 2014), which is given by

$$\mathbf{J}_S = \frac{1}{2} - \log\left(\sqrt{\frac{2}{\pi}}\frac{1}{\sigma}\right)$$

3. Inference

In this section we discuss moments, maximum likelihood estimation, Fisher information matrix and present results of a simulation study.

3.1. *Moment estimators*

Using the first two moments, the moments equations are given by

$$\overline{Z} = \alpha\sigma\kappa_1. \tag{3.1}$$

and

$$\overline{Z^2} = \alpha\sigma^2\kappa_2. \tag{3.2}$$

Solving for σ in equation (3.1), it follows that

$$\sigma = \frac{\overline{Z}}{\alpha\kappa_1}. \tag{3.3}$$

Thus, replacing σ, given in Eq. (3.3), in the Eq. (3.2), it follows that:

$$\overline{Z^2}\alpha\kappa_1^2 - \overline{Z}^2\kappa_2 = 0. \tag{3.4}$$

Solving the Eq. given in (3.2) for α we obtain $\widehat{\alpha}_M$, and hence replacing α by $\widehat{\alpha}_M$ in Eq. (3.3) one obtain $\widehat{\sigma}_M$. This leads to the moments estimators $(\widehat{\sigma}_M, \widehat{\alpha}_M)$ for (σ, α). The Eq. given in (3.4), is solved numerically using function `solve` available in software `MAPLE`.

3.2. *The log-likelihood function*

Let $Z_1,...,Z_n$ be a random sample from random variable $Z \sim PHN(\sigma, \alpha)$. The likelihood function for $\theta = (\sigma, \alpha)$ is $\sum_{i=1}^{n} l(\theta; z_i)$, where $l(\theta; z)$ is the log-likelihood function for θ based on the observation z, such that,

$$l(\theta; z) = \log(2) + \log(\alpha) - \log(\sigma) - \log(\sqrt{2\pi}) - \frac{z^2}{2\sigma^2} + (\alpha-1)\log\left(2\Phi\left(\frac{z}{\sigma}\right) - 1\right).$$

3.3. *Score function*

Some standard algebraic manipulations show that the score function is $\sum_{i=1}^{n} S_\theta(\theta; z_i)$, where $S_\theta(\theta, z) = \partial l(\theta, z)/\partial\theta$ is the vector (S_σ, S_α) with elements

$$S_\sigma = -\frac{1}{\sigma} + \frac{z^2}{\sigma^3} - \frac{2(\alpha-1)}{\sigma^2}\left(\frac{z\phi\left(\frac{z}{\sigma}\right)}{2\Phi\left(\frac{z}{\sigma}\right)-1}\right),$$

$$S_\alpha = \frac{1}{\alpha} + \log[2\Phi\left(\frac{z}{\sigma}\right) - 1].$$

The second derivatives of $l(\theta; z)$ are:

$$\frac{\partial^2 l(\theta;z)}{\partial\sigma^2} = \frac{1}{\sigma^2} - \frac{3z^2}{\sigma^4} + \frac{2(\alpha-1)z\phi\left(\frac{z}{\sigma}\right)}{\sigma^3\left[2\Phi\left(\frac{z}{\sigma}\right)-1\right]}\left(2 - \frac{z^2}{\sigma^2} - \frac{2z\phi\left(\frac{z}{\sigma}\right)}{\sigma(2\Phi\left(\frac{z}{\sigma}\right)-1)}\right),$$

$$\frac{\partial^2 l(\theta;z)}{\partial\alpha^2} = -\frac{1}{\alpha^2},$$

$$\frac{\partial^2 l(\theta;z)}{\partial\sigma\partial\alpha} = -\frac{2z\phi\left(\frac{z}{\sigma}\right)}{\sigma^2(2\Phi\left(\frac{z}{\sigma}\right)-1)}.$$

3.4. *Fisher information matrix*

Using the second derivatives above, the Fisher information matrix for the distribution PHN can be written as

$$I_F(\sigma,\alpha)=\begin{pmatrix} I_{\sigma\sigma} & I_{\sigma\alpha} \\ I_{\sigma\alpha} & I_{\alpha\alpha}, \end{pmatrix},$$

with elements given by

$$I_{\sigma\sigma} = -\frac{1}{\sigma^2} + \frac{3\alpha\kappa_2}{\sigma^2} - \frac{2(\alpha-1)}{\sigma^3}\left[2a_{11} - \frac{a_{31}}{\sigma^2} - \frac{2a_{22}}{\sigma}\right],$$

$$I_{\sigma\alpha} = \frac{1}{\alpha^2},$$

$$I_{\alpha\alpha} = \frac{2}{\sigma^2}a_{11},$$

where $a_{ij} = E\left[Z^i\left(\frac{\phi(Z/\sigma)}{2\Phi(Z/\sigma)-1}\right)^j\right]$ can be computed numerically.

3.5. *Simulation study*

A simple algorithm can be formulated for generating from the PHN distribution.

(i) Simulate $Y \sim U(0,1)$

Table 1. Empirical means and standard deviations for different values of α.

	$n = 20$		$n = 50$		$n = 100$	
α	$\hat{\alpha}(SD)$	RMSEs	$\hat{\alpha}(SD)$	RMSEs	$\hat{\alpha}(SD)$	RMSEs
0.8	0.834(0.188)	0.139	0.814(0.116)	0.093	0.804(0.085)	0.068
1.0	1.054(0.237)	0.186	1.015(0.147)	0.116	1.002(0.103)	0.082
2.0	2.115(0.506)	0.392	2.042(0.312)	0.245	2.025(0.203)	0.162
3.0	3.189(0.744)	0.582	3.084(0.447)	0.353	3.047(0.300)	0.240
5.0	5.285(1.234)	0.954	5.123(0.779)	0.610	5.036(0.505)	0.405

(ii) Compute $X = \sigma\Phi^{-1}\left(\frac{1+Y^{1/\alpha}}{2}\right)$

(iii) Return to (i).

Table 1 shows results of simulations studies, illustrating the behaviour of the MLEs for 1000 generated samples of sizes $n = 20$, 50 and 100 from population distributed as $PHN(1,\alpha)$. For each generated sample, MLEs were computed numerically using a Newton-Raphson procedure. Means, standard deviations (SD) and root mean squared errors (RMSEs) are reported. Observe that as the sample size increases, estimates are closer to the true values and, moreover standard deviations and RMSEs become smaller.

4. Illustration

Hereafter, for illustration purposes, we analyze three data sets. We consider now the computation of the maximum likelihood estimates for the models GHN and PHN based one a real data set.

Description of the data sets.

(1) **I1 Volcanoes data(Table 2):** The real data corresponds to heights (in 100× feet) of 219 volcanoes studied in Tukey (1977). This data set has been recently analyzed in Castillo et al. (2011).

(2) **I2 Survival times(Table 3):** The data analyzed by Kundu et al. (2008) and Leiva et al. (2009) correspond to 72 survival times of guinea pigs injected with different doses of tubercle bacilli.

(3) **I3 Engineering(Table 4):** The real data set analysis using a data set previously analyzed in Birnbaum and Saunders (1969), related to the lifetimes in cycles 10^{-3} of aluminium $6061-T6$ pieces cut in parallel angle with the rotation direction, oscillating at the rate of 18 cycles/s at maximum pressure 31,000psi, with a total sample size of 101 units.

Table 5 presents basic descriptive statistics for the data sets.

Using results from Section 3.1, moments modified estimators used as initial estimates for the maximum likelihood approach.

Table 6 shows parameters estimates by maximum likelihood using the **bbmle** package in program R (2012). For each model we report the value of maximum likelihood estimate and the corresponding Akaike information criterion (AIC), according to Akaike (1976), we consider also

Table 2. Data sets 1: 219 volcanoes heights studied in Tukey (1977)

2	5	6	6	6	7	9	9	10	10	10	11	13	16	16
17	19	19	20	20	21	21	22	22	22	24	24	24	25	25
26	26	26	27	27	28	28	29	29	29	30	31	31	32	32
34	34	35	35	35	35	36	36	36	37	38	39	39	40	41
41	41	42	43	43	43	43	43	44	44	44	46	47	48	48
49	49	49	49	49	50	50	51	51	52	52	52	53	54	54
55	55	56	56	56	56	56	56	57	57	57	59	59	60	60
61	61	64	64	65	65	66	66	66	66	67	67	67	68	68
69	70	70	70	70	71	71	71	72	73	73	74	75	75	75
76	77	78	78	78	79	81	82	82	82	82	83	83	83	85
86	87	89	90	90	90	91	92	93	93	94	94	95	95	96
97	97	99	100	101	101	102	102	103	103	104	104	105	106	108
109	110	111	111	112	113	113	114	116	116	119	121	121	122	124
124	124	125	126	130	133	134	137	138	140	140	156	156	157	162
165	172	179	185	190	193	193	197	199						

Table 3. Data sets 2: survival times of guinea pigs

12	15	22	24	24	32	32	33	34	38	38	43	44	48
52	53	54	54	55	56	57	58	58	59	60	60	60	60
61	62	63	65	65	67	68	70	70	72	73	75	76	76
81	83	84	85	87	91	95	96	98	99	109	110	121	127
129	131	143	146	146	175	175	211	233	258	258	263	297	341
341	376												

Table 4. Data sets 3: 101 observations, maximum stress per cycle 31,000 psi

70	90	96	97	99	100	103	104	104	105	107	108	108	108
109	109	112	112	113	114	114	114	116	119	120	120	120	121
121	123	124	124	124	124	124	128	128	129	129	130	130	130
131	131	131	131	131	132	132	132	133	134	134	134	134	134
136	136	137	138	138	138	139	139	141	141	142	142	142	142
142	142	144	144	145	146	148	148	149	151	151	152	155	156
157	157	157	157	158	159	162	163	163	164	166	166	168	170
174	196	212											

the modified AIC criterion (see, for example, Bozdogan (1987)), typically called consistent AIC (CAIC) and the Bayesian information criterion (BIC) (see, for example, Schwarz (1978)). It can be noticed that AIC, CAIC and BIC show better fit of the PHN model. The models fitted in Castillo et al. (2011) has an AIC approximately 2236.604 for the I1. Fig. 8, 9 and 10 depicts ML fitting

Table 5. Descriptive statistics for the three data sets

Data	Mean	Median	Mode	SD	Variance	Skewness	Kurtosis	Min.	Max.
I1	70.25	65	56	43.01	1850.563	0.84	3.48	2	199
I2	99.82	70	60	81.12	6580.12	1.84	2.89	12	376
I3	133.7	133	142	22.36	499.78	0.326	0.973	70	212

Table 6. MLEs of the model parameters for the three data sets and the corresponding AIC, CAIC and BIC statistics.

Data	Model	σ	α	AIC	CAIC	BIC
I1	PHN	70.084	1.55	2229.835	2229.891	2236.613
	GHN	88.736	1.297	2233.454	2233.509	2240.232
I2	PHN	117.343	1.254	805.291	805.465	809.845
	GHN	129.239	1.017	807.479	807.649	812.028
I3	PHN	55.091	40.639	919.736	919.859	924.966
	GHN	148.06	4.255	943.803	943.925	949.033

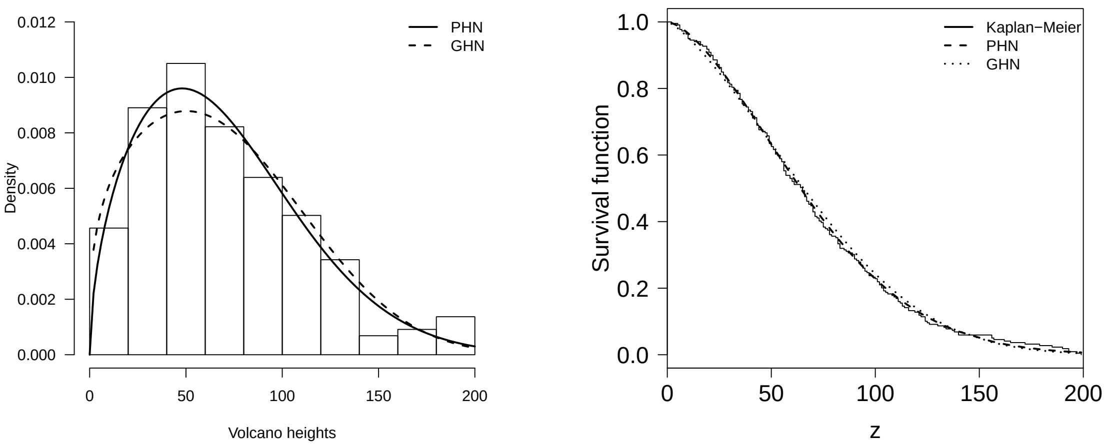

Fig. 8. Histogram for dataset volcano heights, lines represent fitted distributions using maximum likelihood estimators(left) and Survival functions and the empirical survival for I1 (right) .

for both models with the data histogram. In Figures, the empirical survival function with estimated PHN and GHN c.d.f., also shows the good agreement between the PHN model and the I1, I2 and I3 dataset.

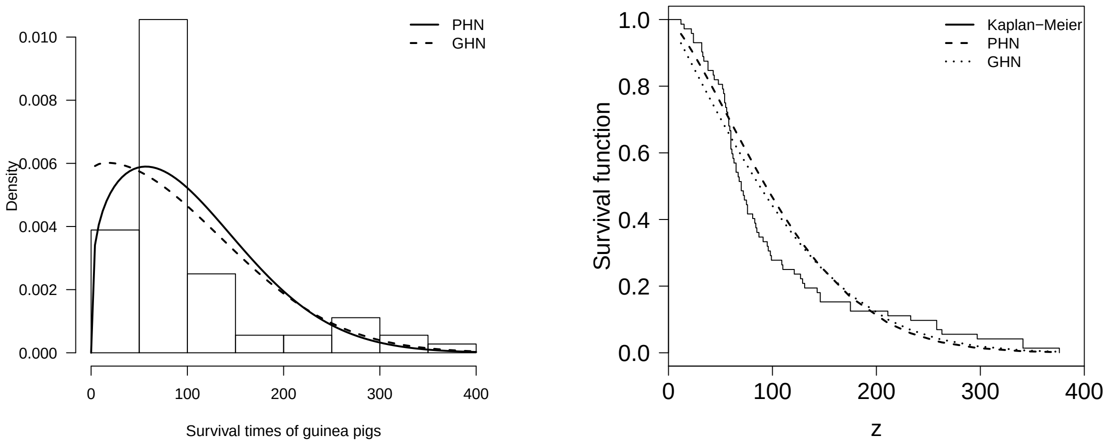

Fig. 9. Histogram for dataset I2, lines represent fitted distributions using maximum likelihood estimators(left) and Survival functions and the empirical survival for I2 (right).

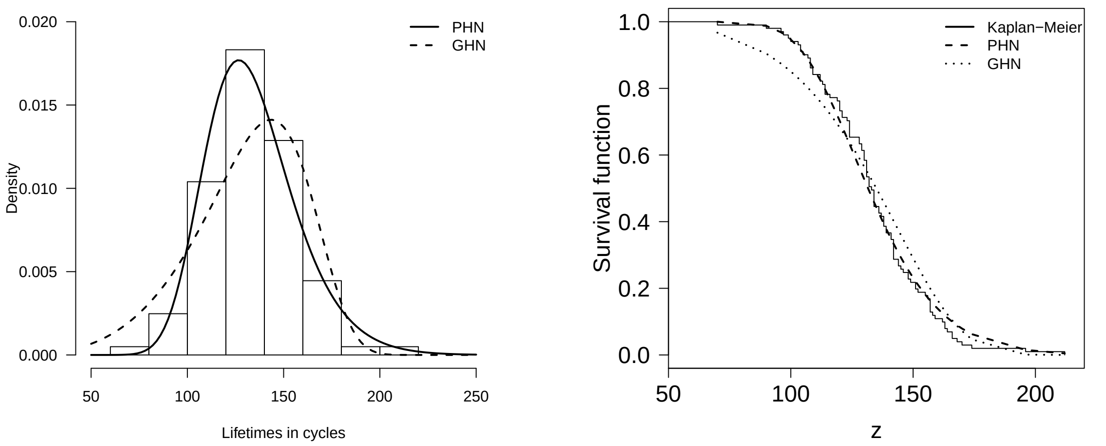

Fig. 10. Histogram for dataset I3, lines represent fitted distributions using maximum likelihood estimates(left) and survival functions and the empirical survival for I3 (right).

Finally, we apply the Cramér-von Mises (W^*) and Anderson-Darling (A^*) test statistics. The W* and A* test statistics are described in details in Chen and Balakrishnan (1995). In general, the smaller the values of W^* and A^* statistics, the better the fit to the data. The values of these statistics for all models are given in Table 7. As expected, the values of W^* and A^* for the PHN model fits better than the GHN model for the data analyzed.

Table 7. Goodness-of-fit tests.

Data	Model	Statistics	
		W*	A*
I1	PHN	0.017	0.172
	GHN	0.059	0.469
I2	PHN	0.570	3.145
	GHN	0.580	3.197
I3	PHN	0.106	0.602
	GHN	0.263	1.684

5. Concluding remarks

This paper focuses on studying a submodel of the family of models introduced in Pescim et al. (2010). This model has two parameters and is an alternative to the generalized half-normal (GHN) studied in Cooray and Ananda (2008). Most of the results present explicit expressions which are easily computable. A simulation study is conducted for the shape parameter and show that the MLE present small bias for small and moderate sample sizes. In the application, Akaike's AIC, CAIC and BIC criterion is used for model comparison which shows that the proposed model presents better fit to the data sets analyzed than the model proposed in Cooray and Ananda (2008).

Acknowledgments

We acknowledge two referees and the Editor G.G. Hamedani for comments and suggestions that substantially improved the presentation. The research of Yolanda M. Gómez was supported by Becas-Chile of the Chilean government. The research of Heleno Bolfarine was supported by CNPq and Fapesp funded by the Brasilian government.

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Published by Atlantis Press