

## Dynamic Cumulative Residual and Past Inaccuracy Measures

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In this paper, we have developed measures of dynamic cumulative residual and past inaccuracy. We have studied characterization results under proportional hazard model in case of dynamic cumulative residual inaccuracy and under proportional reversed hazard model in case of dynamic cumulative past inaccuracy measure. We have characterized certain specific lifetime distributions using the measures proposed. Some generalized results have also been considered.

*Keywords:* Shannon entropy; Cumulative residual entropy; Cumulative inaccuracy measure; Mean residual life function; Proportional hazard model.

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### 1. Introduction

Let  $X$  and  $Y$  be two non-negative random variables representing time to failure of two systems with p.d.f. respectively  $f(x)$  and  $g(x)$ . Let  $F(x) = P(X \leq x)$  and  $G(y) = P(Y \leq y)$  be failure distributions, and  $\bar{F}(x) = 1 - F(x)$ ,  $\bar{G}(x) = 1 - G(x)$  be survival functions of  $X$  and  $Y$  respectively. Shannon's (1948) measure of uncertainty associated with the random variable  $X$  and Kerridge measure of inaccuracy (1961) are given as

$$H(f) = - \int_0^{\infty} f(x) \log f(x) dx, \quad (1.1)$$

and

$$H(f;g) = - \int_0^{\infty} f(x) \log g(x) dx, \quad (1.2)$$

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respectively. In case  $g(x) = f(x)$ , then (1.2) reduces to (1.1).

The measures (1.1) and (1.2) are not applicable to a system which has survived for some unit of time, say  $t$ . Ebrahimi (1996) considered the entropy of the residual lifetime  $X_t = [X - t | X > t]$  as a dynamic measure of uncertainty given by

$$H(f; t) = - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx. \tag{1.3}$$

Extending the dynamic measure of information, a dynamic measure of inaccuracy, refer to Taneja et al. (2009) is given as

$$H(f, g; t) = - \int_t^\infty \frac{f(x)}{\bar{F}(t)} \log \frac{g(x)}{\bar{G}(t)} dx. \tag{1.4}$$

Rao et al. (2004) introduced an alternate measure of entropy called *cumulative residual entropy (CRE)* of a random variable  $X$  defined as

$$\xi(F) = - \int_0^\infty \bar{F}(x) \log \bar{F}(x) dx, \tag{1.5}$$

where  $\bar{F}(x) = 1 - F(x)$  is the survival function of  $X$ .

This measure is based on cumulative distribution function (CDF) rather than probability density function, and is thus, in general more stable since the distribution function is more regular because it is defined in an integral form unlike the density function which is defined as the derivative of the distribution function. Some general results regarding this measure have been studied by Rao (2005), Drissi et al. (2008) and Navarro et al. (2010).

Asadi and Zohrevand (2007) have defined the dynamic cumulative residual entropy (DCRE) as the cumulative residual entropy of the residual lifetime  $X_t = [X - t | X > t]$ . This is given by

$$\xi(F; t) = - \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log \frac{\bar{F}(x)}{\bar{F}(t)} dx. \tag{1.6}$$

In this paper we propose dynamic cumulative residual and past inaccuracy measures and study their characterization results. The paper is organized as follows. In Section 2, we consider cumulative residual inaccuracy measure and derive a lower bound to it. Section 3 contains the dynamic cumulative residual inaccuracy. In Section 4, we study characterization results concerning dynamic cumulative residual inaccuracy measure and also characterize a few specific lifetime distributions. Section 5 considers dynamic cumulative past inaccuracy measure and its characterization result. Finally we give some conclusions and comments.

## 2. Cumulative Residual Inaccuracy Measure

If  $\bar{F}(\cdot)$  and  $\bar{G}(\cdot)$  are survival functions of lifetime random variables  $X$  and  $Y$  respectively, then the cumulative residual inaccuracy is defined as

$$\xi(F; G) = - \int_0^\infty \bar{F}(x) \log \bar{G}(x) dx. \tag{2.1}$$

Here and throughout this communication, we consider the random variables  $X$  and  $Y$  with the same support.

When the two distributions  $F$  and  $G$  coincide, the measure (2.1) reduces to the cumulative residual entropy (1.5). Even if the two random variables  $X$  and  $Y$  satisfy the proportional hazard model (PHM), refer to Cox (1972) and Efron (1981), that is, if  $\lambda_G(x) = \beta \lambda_F(x)$ , or equivalently

$$\bar{G}(x) = [\bar{F}(x)]^\beta, \tag{2.2}$$

for some constant  $\beta > 0$ , then obviously the cumulative residual inaccuracy (2.1) reduces to a constant multiple of the cumulative residual entropy (1.5).

**Example 2.1** Let a non-negative random variable  $X$  be uniformly distributed over  $(a, b)$ ,  $a < b$ , with density and distribution functions respectively given by

$$f(x) = \frac{1}{b-a} \text{ and } F(x) = \frac{x-a}{b-a}, \quad a < x < b.$$

If the random variables  $X$  and  $Y$  satisfy the proportional hazard model (PHM), then the distribution function of the random variable  $Y$  is

$$\bar{G}(x) = [\bar{F}(x)]^\beta = \left[ \frac{b-x}{b-a} \right]^\beta, \quad a < x < b, \quad \beta > 0.$$

Substituting these in (2.1) and simplifying we obtain the cumulative inaccuracy measure as

$$\xi(F; G) = \frac{\beta(b-a)}{4}.$$

**2.1. A lower bound to  $\xi(F; G)$**

Before deriving the lower bound to  $\xi(F; G)$ , we define the *log-sum inequality* given as follows: Let  $m$  be a sigma finite measure. If  $f$  and  $g$  are positive and  $m$ -integrable, then

$$\int \log \left( \frac{f}{g} \right) dm \geq \left[ \int f dm \right] \log \frac{\int f dm}{\int g dm}. \tag{2.3}$$

We prove the following result.

**Theorem 2.1.** *If  $X$  and  $Y$  are non-negative random variables with finite means  $E(X)$  and  $E(Y)$  respectively and CRE  $\xi(F)$  is finite, then*

$$\xi(F; G) \geq \int_0^\infty F(x)\bar{F}(x)dx + E(X) - E(Y). \tag{2.4}$$

**Proof** We have

$$\begin{aligned} \xi(F; G) &= - \int_0^\infty \bar{F}(x) \log \bar{G}(x) dx, \\ &= - \int_0^\infty \bar{F}(x) \log \bar{F}(x) dx + \int_0^\infty \bar{F}(x) \log \frac{\bar{F}(x)}{\bar{G}(x)} dx. \end{aligned}$$

Using the log-sum inequality (2.3), we have

$$\begin{aligned} \xi(F; G) &\geq \xi(F) + \left( \int_0^\infty \bar{F}(x) dx \right) \log \frac{\int_0^\infty \bar{F}(x) dx}{\int_0^\infty \bar{G}(x) dx}, \\ &\geq \xi(F) + E(X) \log \frac{E(X)}{E(Y)}. \end{aligned} \tag{2.5}$$

Next, using the inequality  $x \log \frac{x}{y} \geq x - y$  for non-negative  $x$  and  $y$  in (2.5), we obtain

$$\begin{aligned} \xi(F; G) &\geq \int_0^\infty \bar{F}(x) F(x) dx + E(X) \log \frac{E(X)}{E(Y)}, \\ &\geq \int_0^\infty \bar{F}(x) F(x) dx + E(X) - E(Y). \end{aligned}$$

This proves the result.

### 3. Dynamic Cumulative Residual Inaccuracy

In life-testing experiments normally the experimenter has information about the current age of the system under consideration. Obviously the cumulative residual inaccuracy measure (2.1) defined above is not suitable in such a situation and should be modified to take into account the current age also. Further, if  $X$  is the lifetime of a component, which has survived upto time  $t$ , then the random variable  $X_t = [X - t | X > t]$ , called the residual lifetime random variable, has the survival function

$$\bar{F}_t(x) = \begin{cases} \frac{\bar{F}(x)}{\bar{F}(t)} & ; \text{ if } x > t \\ 1 & ; \text{ otherwise} \end{cases}$$

and similarly for  $\bar{G}_t(x)$ . Further we know that the mean residual life function  $\delta_F(t) = E[X - t | X > t] = \frac{\int_t^\infty \bar{F}(x) dx}{\bar{F}(t)}$  and the hazard rate  $\lambda_F(t) = \frac{f(t)}{\bar{F}(t)}$  characterize the distribution function  $F(\cdot)$ , and the relation between the two is given by

$$\lambda_F(t) = \frac{1 + \delta'_F(t)}{\delta_F(t)}, \tag{3.1}$$

where  $\delta'_F(t) = \frac{d}{dt} \delta_F(t)$ .

Analogous to the measure (2.1) the cumulative inaccuracy measure for the residual lifetime  $X_t = [X - t | X > t]$ , is given by

$$\begin{aligned} \xi(F, G; t) &= - \int_t^\infty \bar{F}_t(x) \log \bar{G}_t(x) dx, \\ &= - \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log \frac{\bar{G}(x)}{\bar{G}(t)} dx. \end{aligned} \tag{3.2}$$

We define the measure (3.2) as the *dynamic cumulative residual inaccuracy* measure. Obviously when  $t = 0$ , then (3.2) becomes (2.1).

**Example 3.1** Let  $X$  be a non-negative random variable with p.d.f.

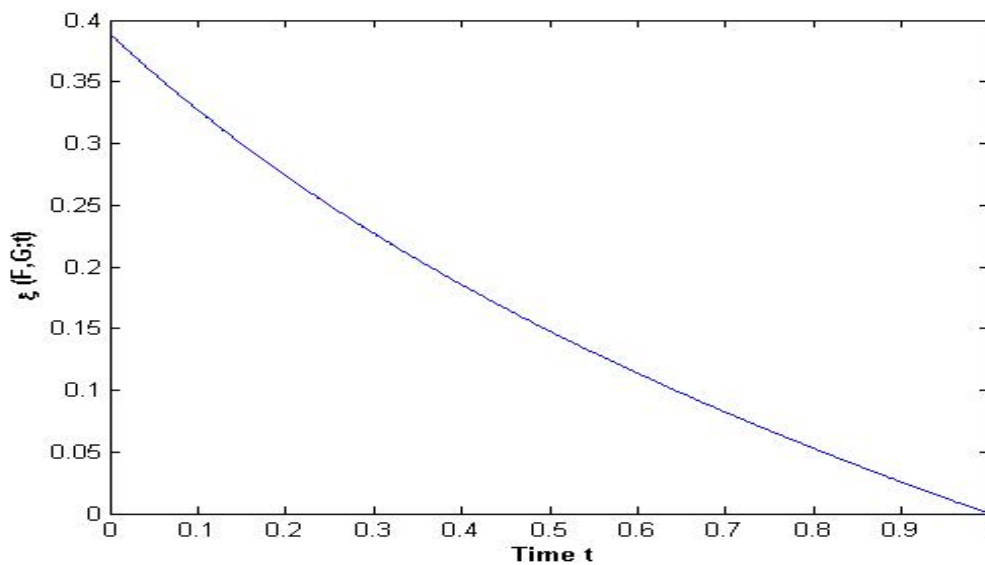
$$f_X(x) = \begin{cases} 2x & ; \text{ if } 0 \leq x < 1 \\ 0 & ; \text{ otherwise} \end{cases}$$

and the survival function  $\bar{F}(x) = 1 - F(x) = (1 - x^2)$ , and let the random variable  $Y$  be uniformly distributed over  $(0, 1)$  with density and survival functions given respectively by  $g_Y(x) = 1$  and  $\bar{G}_Y(x) = 1 - x$ ,  $0 < x < 1$ .

Substituting these values in (3.2), we obtain the dynamic cumulative residual inaccuracy measure as

$$\xi(F, G; t) = \begin{cases} \frac{9(1-t) - 2(1-t)^2}{18(1+t)} & ; \text{ if } 0 \leq t < 1 \\ 0 & ; \text{ otherwise} \end{cases}$$

The behaviour of the dynamic cumulative residual inaccuracy measure  $\xi(F, G; t)$  for  $t \in (0, 1)$  is shown in Fig. 3.1.



**Fig. 3.1:** Plot of  $\xi(F, G; t)$  against  $t \in [0, 1]$ .

#### 4. Characterization Problem

The general characterization problem is to determine when the proposed dynamic cumulative residual inaccuracy measure (3.2) characterizes the distribution function uniquely. We study the characterization problem under the proportional hazard model (2.2).

**Theorem 4.1.** Let  $X$  and  $Y$  be two non negative random variables with survival functions  $\bar{F}(x)$  and  $\bar{G}(x)$  satisfying the proportional hazard model (2.2). Let  $\xi(F, G; t) < \infty, \forall t \geq 0$  be an increasing function of  $t$ , then  $\xi(F, G; t)$  uniquely determines the survival function  $\bar{F}(x)$  of the variable  $X$ .

**Proof** Rewriting (3.2) as

$$\xi(F, G; t) = -\frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x) \log \bar{G}(x) dx + \delta_F(t) \log \bar{G}(t), \tag{4.1}$$

where  $\delta_F(t)$  is the mean residual life function. Substituting (2.2) into (4.1) gives

$$\xi(F, G; t) = -\frac{\beta}{\bar{F}(t)} \int_t^\infty \bar{F}(x) \log \bar{F}(x) dx + \beta \delta_F(t) \log \bar{F}(t).$$

Differentiating this w.r.t.  $t$  both sides, we obtain

$$\xi'(F, G; t) = \beta \log \bar{F}(t) [1 + \delta_F'(t)] - \beta \lambda_F(t) \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)} \log \bar{F}(x) dx - \beta \lambda_F(t) \delta_F(t), \tag{4.2}$$

where  $\lambda_F(t)$  is hazard rate function. Substituting (3.1) and (4.1) in (4.2) we obtain

$$\xi'(F, G; t) = \lambda_F(t) \{ \xi(F, G; t) - \beta \delta_F(t) \}. \tag{4.3}$$

Let  $F_1, G_1$  and  $F_2, G_2$  be two sets of the probability distribution functions satisfying the proportional hazard model, that is,  $\lambda_{G_1}(x) = \beta \lambda_{F_1}(x)$ , and  $\lambda_{G_2}(x) = \beta \lambda_{F_2}(x)$ , and let

$$\xi(F_1, G_1; t) = \xi(F_2, G_2; t) \quad \forall t \geq 0. \tag{4.4}$$

Differentiating it both sides w.r.t.  $t$ , and using (4.3), we obtain

$$\lambda_{F_1}(t) \{ \xi(F_1, G_1; t) - \beta \delta_{F_1}(t) \} = \lambda_{F_2}(t) \{ \xi(F_2, G_2; t) - \beta \delta_{F_2}(t) \}. \tag{4.5}$$

If for all  $t \geq 0$ ,  $\lambda_{F_1}(t) = \lambda_{F_2}(t)$ , then  $\bar{F}_1(t) = \bar{F}_2(t)$  and the proof will be over, otherwise, let

$$A = \{ t : t \geq 0, \text{ and } \lambda_{F_1}(t) \neq \lambda_{F_2}(t) \} \tag{4.6}$$

and assume the set  $A$  to be non empty . Thus for some  $t_0 \in A$ ,  $\lambda_{F_1}(t_0) \neq \lambda_{F_2}(t_0)$ . Without loss of generality suppose that  $\lambda_{F_2}(t_0) > \lambda_{F_1}(t_0)$ . Using this, (4.5) for  $t = t_0$  gives

$$\xi(F_1, G_1; t_0) - \beta \delta_{F_1}(t_0) > \xi(F_2, G_2; t_0) - \beta \delta_{F_2}(t_0),$$

which implies that

$$\delta_{F_1}(t_0) < \delta_{F_2}(t_0),$$

a contradiction. Thus the set  $A$  is empty set and this concludes the proof.

In the next result, based on dynamic cumulative residual inaccuracy measure (3.2), we characterize some specific lifetime distributions.

**Theorem 4.2.** Let  $X$  and  $Y$  be two non-negative continuous random variables satisfying the proportional hazard model (2.2). If  $X$  is with mean residual life  $\delta_F(t)$ , then the dynamic cumulative residual inaccuracy measure

$$\xi(F, G; t) = c \delta_F(t), \quad c > 0 \tag{4.7}$$

if, and only if

- (i)  $X$  follows the exponential distribution for  $c = \beta$ ,
- (ii)  $X$  follows the Pareto distribution for  $c > \beta$ ,
- (iii)  $X$  follows the finite range distribution for  $0 < c < \beta$ .

**Proof** First we prove the 'if' part.

(i) If  $X$  has an exponential distribution with survival function  $\bar{F}(x) = \exp(-\theta x)$ ,  $\theta > 0$ , then the mean residual life function  $\delta_F(t) = \frac{1}{\theta}$ . The dynamic cumulative residual inaccuracy measure (3.2) under PHM (2.2) is given as

$$\xi(F, G; t) = \frac{\beta}{\theta} = c \delta_F(t),$$

for  $c = \beta$ .

(ii) If  $X$  follows a Pareto distribution with p.d.f.

$$f(x) = \frac{ab^a}{(x+b)^{a+1}}, \quad a > 1, \quad b > 0,$$

then the survival function is

$$\bar{F}(x) = 1 - F(x) = \left(1 + \frac{x}{b}\right)^{-a} = \frac{b^a}{(x+b)^a},$$

and the mean residual life is

$$\delta_F(t) = \frac{\int_t^\infty \bar{F}(x) dx}{\bar{F}(t)} = \frac{t+b}{a-1}. \tag{4.8}$$

The dynamic cumulative inaccuracy measure (3.2), under PHM (2.2) is given by

$$\xi(F, G; t) = \frac{\beta a(t+b)}{(a-1)^2} = c \delta_F(t),$$

for  $c = \frac{\beta a}{a-1} > \beta$ .

(iii) In case  $X$  follows a finite range distribution with p.d.f.

$$f(x) = a(1-x)^{a-1}, \quad a > 1, \quad 0 \leq x \leq 1,$$

then the survival function is

$$\bar{F}(x) = 1 - F(x) = (1-x)^a,$$

and the mean residual life is

$$\delta_F(t) = \frac{1-t}{a+1}.$$

The inaccuracy measure (3.2) under PHM (2.2) is given by

$$\xi(F, G; t) = \frac{\beta a(1-t)}{(a+1)^2} = c\delta_F(t),$$

for  $c = \frac{\beta a}{a+1} < \beta$ .

This proves the 'if' part.

To prove the 'only if' part, consider (4.7) to be valid. Using (4.1) under PHM, it gives

$$-\frac{\beta}{\bar{F}(t)} \int_t^\infty \bar{F}(x) \log \bar{F}(x) dx + \beta \delta_F(t) \log \bar{F}(t) = c\delta_F(t).$$

Differentiating it both sides w.r.t.  $t$ , we obtain

$$\begin{aligned} \frac{c}{\beta} \delta_F'(t) &= \delta_F'(t) \log \bar{F}(t) - \lambda_F(t) \delta_F(t) + \log \bar{F}(t) \\ &\quad - \lambda_F(t) \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x) \log \bar{F}(x) dx \\ &= \delta_F'(t) \log \bar{F}(t) - \lambda_F(t) \delta_F(t) + \log \bar{F}(t) + \lambda_F(t) \left[ \frac{c}{\beta} \delta_F(t) - \delta_F(t) \log \bar{F}(t) \right]. \end{aligned}$$

From (3.2) put  $\delta_F'(t) = \lambda_F(t) \delta_F(t) - 1$  and simplify, we obtain

$$\lambda_F(t) \delta_F(t) = \frac{c}{\beta},$$

which implies

$$\delta_F'(t) = \frac{c}{\beta} - 1.$$

Integrating both sides of this w.r.t.  $t$  over  $(0, x)$  yields

$$\delta_F(x) = \left( \frac{c}{\beta} - 1 \right) x + \delta_F(0). \tag{4.9}$$

The mean residual life function  $\delta_F(x)$  of a continuous non-negative random variable  $X$  is linear of the form (4.9) if, and only if the underlying distribution is exponential for  $c = \beta$ , Pareto for  $c > \beta$ , or finite range for  $0 < c < \beta$ , refer to Hall and Wellner (1981). This completes the theorem.



Theorem 4.2 can be extended by taking  $c$  as a function of  $t$ . We state the following result:

**Theorem 4.3.** *Let  $X$  and  $Y$  be two non-negative continuous random variables satisfying the proportional hazard model (2.2). If*

$$\xi(F, G; t) = c(t)\delta_F(t), \text{ for } t \geq 0, \tag{4.10}$$

then

$$\delta_F(t) = \left[ k + \left( \int_0^t \left\{ \frac{c(x) - \beta}{\beta} \right\} e^{\frac{c(x)}{\beta}} dx \right) \right] e^{-\frac{c(t)}{\beta}}, \tag{4.11}$$

where  $k = \delta_F(0)e^{\frac{c(0)}{\beta}}$ .

**Proof** Substituting (4.10) in (4.3), we obtain

$$\xi'(F, G; t) = \lambda_F(t)\delta_F(t)\{c(t) - \beta\}. \tag{4.12}$$

Differentiating (4.10) w.r.t.  $t$  and substituting from (4.12), we obtain

$$c'(t)\delta_F(t) + c(t)\delta_F'(t) = \lambda_F(t)\delta_F(t)\{c(t) - \beta\}.$$

Substituting  $\lambda_F(t)\delta_F(t) = 1 + \delta_F'(t)$  from (3.2) and simplifying, we obtain

$$\delta_F'(t) + \frac{c'(t)}{\beta}\delta_F(t) = \frac{c(t) - \beta}{\beta}, \tag{4.13}$$

a linear differential equation in  $\delta_F(t)$ . Solving this we obtain (4.10).

For  $c(t) = at + b$ ,  $t > 0$  and  $a > 0$ , from (4.10), we obtain the general model with mean residual life function

$$\delta_F(t) = ke^{-\frac{(at+b)}{\beta}} + \frac{at - 2\beta + b}{a} - \frac{(b - 2\beta)e^{-\frac{at}{\beta}}}{a}. \tag{4.14}$$

If  $a = 0$ , we obtain the characterization results given by Theorem 4.2.

Further for  $\beta = 1$ , (4.14) reduces to

$$\delta_F(t) = ke^{-at-b} + \frac{b - 2 + at}{a} - \frac{(b - 2)e^{-at}}{a},$$

a result given by Navarro J. et al. (2010) in context with the cumulative residual entropy (1.3).

### 5. Dynamic Cumulative Past Inaccuracy Measure

Measures of uncertainty in context with past lifetime distributions have been studied extensively in the literature, refer to, Di Crescenzo and Longobardi (2002, 2004) Nanda and Paul (2006), Kumar et al. (2010). For instance if at time  $t$ , a system which is observed only at certain preassigned inspection times, is found to be down, then the uncertainty of the system's life relies on the past, that is, at which instant in  $(0, t)$  the system has failed. In this situation, the random variable  ${}_tX = [X|X \leq t]$  is suitable to describe the time elapsed between the failure of a system and the time when it is found

to be 'down'. The past lifetime random variable  ${}_tX$  is related with two relevant ageing functions, the reversed hazard rate defined by  $\mu_F(x) = \frac{f(x)}{F(x)}$ , and the mean past lifetime (MPT) defined by  $\delta_F^*(t) = E(t - X | X < t) = \frac{1}{F(t)} \int_0^t F(x) dx$ , which are further related as follows

$$\mu_F(t) = \frac{1 - \delta_F^{t*}(t)}{\delta_F^*(t)}, \tag{5.1}$$

where  $\delta_F^{t*}(t) = \frac{d}{dt} \delta_F^*(t)$ . For further results on reversed hazard rate function refer to Gupta and Nanda (2001).

In analogy with the cumulative residual entropy (CRE) measure (2.1), Di Crescenzo and Longobardi (2009) introduced and studied the cumulative entropy, defined as

$$\xi^*(F) = - \int_0^\infty F(x) \log F(x) dx. \tag{5.2}$$

A dynamic version of the cumulative entropy (5.2) given as

$$\xi^*(F; t) = - \int_0^t \frac{F(x)}{F(t)} \log \frac{F(x)}{F(t)} dx, \tag{5.3}$$

was also studied by Di Crescenzo and Longobardi (2009).

Analogous to the Kerridge measure of inaccuracy (1.2), we propose a cumulative inaccuracy measure as

$$\xi^*(F; G) = - \int_0^\infty F(x) \log G(x) dx, \tag{5.4}$$

where  $F(x)$  is the baseline distribution function and  $G(x)$  can be considered as some reference distribution function. When these two distributions coincide, the measure (5.4) reduces to the measure (5.2) the cumulative entropy.

In case the two random variables  $X$  and  $Y$  satisfy the *proportional reversed hazard model* (PRHM), refer to Gupta et al. (2007), that is, if  $\mu_G(x) = \beta \mu_F(x)$ , or equivalently

$$G(x) = [F(x)]^\beta, \beta > 0, \tag{5.5}$$

then obviously the cumulative inaccuracy measure (5.4) reduces to a constant multiple of the cumulative information measure (5.2).

The distribution function of the past lifetime random variable  $[X | X \leq t]$  is given by

$$F_{tX}(x) = \begin{cases} \frac{F(x)}{F(t)} & ; \text{ if } x < t \\ 1 & ; \text{ otherwise} \end{cases}$$

and similarly for  $\bar{G}_t(x)$ . Thus the cumulative inaccuracy measure analogous to the inaccuracy measure (5.4), for the past lifetime distribution is given by

$$\begin{aligned} \xi^*(F, G; t) &= - \int_0^t F_{tX}(x) \log G_{tX}(x) dx, \\ &= - \int_0^t \frac{F(x)}{F(t)} \log \frac{G(x)}{G(t)} dx. \end{aligned} \tag{5.6}$$

We define the measure (5.6) as the *dynamic cumulative past inaccuracy measure*. When  $t \rightarrow \infty$ , the measure (5.6) reduces to (5.4).

**Example 5.1** Let  $X$  and  $Y$  be two nonnegative random variables having distribution functions respectively

$$F(x) = \begin{cases} \frac{x^2}{2}, & \text{for } 0 \leq x < 1 \\ \frac{x^2+2}{6}, & \text{for } 1 \leq x < 2 \\ 1 & \text{for } x \geq 2 \end{cases}$$

and

$$G(x) = \begin{cases} \frac{x^2+x}{4}, & \text{for } 0 \leq x < 1 \\ \frac{x}{2}, & \text{for } 1 \leq x < 2 \\ 1 & \text{for } x \geq 2 \end{cases}$$

The cumulative past inaccuracy measure is given by

$$\xi^*(F, G; t) = \begin{cases} \frac{2t}{9} - \frac{(t-2)}{6t} - \frac{1}{3t^2} \log(t+1), & \text{for } 0 < t < 1 \\ \frac{t}{9} + \frac{16t}{9(t^2+2)} - \frac{17}{18(t^2+2)} - \frac{18 \log 2 + 24 \log t}{18(t^2+2)}, & \text{for } 1 \leq t < 2 \\ \log 2 + \frac{1}{6} \log 5 - \frac{41}{54} - \frac{8}{3} \tan^{-1}\left(\frac{1}{2}\right) & \text{for } t \geq 2 \end{cases}$$

Analogous to Theorem 4.1, the characterization problem in case of the dynamic measure (5.6) under the proportional reversed hazard rate model (5.5) is given as follows.

**Theorem 5.1.** *Let  $X$  and  $Y$  be two non-negative random variables with distribution functions  $F(\cdot)$  and  $G(\cdot)$  satisfying the proportional reversed hazard rate model (5.5). Let  $\xi^*(F, G; t) < \infty, \forall t \geq 0$  be an decreasing function of  $t$ , then  $\xi^*(F, G; t)$  uniquely determines the distribution function  $F(\cdot)$  of the variable  $X$ .*

The proof is silmilar to that of Theorem 4.1 . Hence omitted.

Next, we characterize a specific distribution by using the dynamic cumulative past inaccuracy measure. The result is stated as follows.

**Theorem 5.2.** *If  $F(\cdot)$  and  $G(\cdot)$  are two distribution functions satisfying the proportional reversed hazard model (5.5), then the dynamic cumulative past inaccuracy measure*

$$\xi^*(F, G; t) = c\delta_F^*(t) , 0 < c < \beta . \tag{5.7}$$

if, and only if  $t F(x) = \left(\frac{x}{b}\right)^{\frac{c}{\beta-c}} , b > 0 .$

**Proof** Rewriting (5.6) as

$$\xi^*(F, G; t) = -\frac{1}{F(t)} \int_0^t F(x) \log G(x) dx + \delta_F^*(t) \log G(t) , \tag{5.8}$$

Substituting (5.5), this gives

$$\xi^*(F, G; t) = -\frac{\beta}{F(t)} \int_0^t F(x) \log F(x) dx + \beta \delta_F^*(t) \log F(t) . \tag{5.9}$$

Differentiating this w.r.t.  $t$  both sides, we obtain

$$\xi'^*(F, G; t) = \beta \log F(t) [\delta_F'^*(t) - 1] + \beta \mu_F(t) \int_0^t \frac{F(x)}{F(t)} \log F(x) dx + \beta \mu_F(t) \delta_F^*(t), \quad (5.10)$$

Substituting (5.1) and (5.9) in Eq. (5.10), we obtain

$$\xi'^*(F, G; t) = \mu_F(t) \{ \beta \delta_F^*(t) - \xi^*(F, G; t) \}. \quad (5.11)$$

Let us take that (5.7) is valid, then differentiate both side w.r.t.  $t$ , we get

$$\xi'^*(F, G; t) = c \delta_F'^*(t). \quad (5.12)$$

Put these value to (5.11), we get

$$c \delta_F'^*(t) = (\beta - c) \mu_F(t) \delta_F^*(t). \quad (5.13)$$

Using (5.1) and simplify, we obtain

$$\delta_F'^*(t) = \left( \frac{\beta - c}{\beta} \right) = 1 - \frac{c}{\beta}. \quad (5.14)$$

This gives

$$\delta_F^*(t) = \left( \frac{\beta - c}{\beta} \right) t. \quad (5.15)$$

Divide (5.14) by (5.15), we obtain

$$\frac{1 - \delta_F'^*(t)}{\delta_F^*(t)} = \mu_F(t) = \left( \frac{c}{\beta - c} \right) \frac{1}{t}. \quad (5.16)$$

We know the relationship between reversed hazard rate and distribution function is given by

$$F(x) = \exp \left[ \int_0^x \mu_F(t) dt \right],$$

this gives

$$F(x) = \left( \frac{x}{b} \right)^{\frac{c}{\beta - c}}, \quad b > 0. \quad (5.17)$$

The reverse part is straightforward and easy to prove.

**Example 5.2** Let  $X$  and  $Y$  be two non-negative random variables satisfying the proportional reversed hazard model (PRHM) and let

$$f_X(x) = \begin{cases} ax^{a-1} & ; \text{ if } 0 \leq x < 1, a > 0 \\ 0 & ; \text{ otherwise} \end{cases}$$

The distribution function  $F(x) = x^a$ , and  $G(x) = [F(x)]^\beta$ ,  $\beta > 0$ .

Substituting these values in (5.6), after simplification we get

$$\xi^*(F, G; t) = \frac{t}{(a+1)^2} = c \delta_F^*(t),$$

where  $c = \frac{1}{a+1}$  and mean past lifetime is  $\delta_F^*(t) = \frac{t}{a+1}$ .

Next, we extend the result (5.7) to a general case taking  $c$  as a function of  $t$ . We state the following result:

**Theorem 5.3.** *If  $X$  and  $Y$  satisfy the PRHM (5.5) and*

$$\xi^*(F, G; t) = c(t)\delta_F^*(t), \text{ for } t \geq 0, \tag{5.18}$$

then

$$\delta_F^*(t) = \left( \int_0^t \left\{ \frac{\beta - c(x)}{\beta} \right\} e^{\frac{c(x)}{\beta}} dx \right) e^{-\frac{c(t)}{\beta}}. \tag{5.19}$$

The proof is similar to that of Theorem 4.3. Hence omitted.

For  $c(t) = at + b, t > 0$  and  $a > 0$ , from (5.19), we obtain the general model with mean past lifetime function

$$\delta_F^*(t) = \frac{2\beta - at - b}{a} + \frac{(b - 2\beta)e^{-\frac{at}{\beta}}}{a}. \tag{5.20}$$

Further for  $\beta = 1$ , (5.20) reduces to

$$\delta_F^*(t) = \frac{2 - at - b}{a} - \frac{(b - 2)e^{-at}}{a},$$

a result given by Di Crescenzo and Longobardi (2009) in context with cumulative entropy.

**Conclusions and Comments:** The cumulative distribution function based measures of entropy  $\xi(F)$  and  $\xi^*(F)$  are in general more stable in comparison to probability density function based measure  $H(f)$  given by Shannon (1948). The concept of cumulative entropy is extended to cumulative inaccuracy and further to their dynamic versions viz. cumulative residual inaccuracy  $\xi(F, G; t)$  and cumulative past inaccuracy  $\xi^*(F, G; t)$ . The characterization results concerning, when these inaccuracy measures determine the underlying distributions uniquely, have been studied and a few specific lifetime distributions have been characterized. It is expected that dynamic inaccuracy measures introduced in this paper will encourage the researchers to explore this concept further.

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