Gaussian Curvature Connection of Bézier Surfaces

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Abstract: Based on the properties of Gaussian curvature connection and the theory of differential geometry, a sufficient condition for Gaussian curvature connection between two adjacent Bézier surfaces is obtained. Next, a new method called Gaussian curvature connection is put forward by using the condition. The connection result is better than the connection with continuity of tangent plane, but the condition of connection is weaker than that of curvature connection.

Introduction

Smooth connection of surfaces is important contents in the computer aided design and computer aided manufacturing. Generally speaking, surfaces in real life are not analytic ones, they are usually made by piecing several surface patches together, and thus the problems arise on how to make the surfaces smooth. Now three methods of smooth connection of surface patches are often used, which are connection, connection with continuity of tangent plane (G^1continuity) and connection with curvature continuity (G^2continiuty). Some authors have studied these problems, Kahmann [1] presented the condition for G^1 continuity connection of two Bézier surfaces along their common boundary $\bar{r}(\bar{u},0) = r(u,1)(\bar{u} = u)$ as follows

$$\Delta^{0,1} b_{i,0} = a\Delta^{0,1} a^k_{i,n-1} + b\,\frac{m-i}{n}\Delta^{1,0} a^k_{i,n} + g\,\frac{i}{n}\Delta^{1,0} a^i_{i-1,n},\ i = 1,2,\ldots,m.$$

Where

$$r(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{i,j} B_{i,m}(u) B_{j,n}(v),\ \bar{r}(\bar{u},\bar{v}) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j} B_{i,m}(\bar{u}) B_{j,n}(\bar{v}).$$

Du and Schmitt [2] put forward the method of G^1 connection of Bézier surfaces patches around a common vertex. Meng [3] once discussed curvature connection of triangular Bézier patches with a common vertex, Zhang [4] studied G^1 connection of polynomial surfaces around a corner. Li and Liu [5] presented a method in search of curvature continuous conditions. Jones [6] gave the corresponding method. Shi has introduced and studied some approaches to constructing actual surfaces with Bézier surfaces. In this paper a new method called Gaussian curvature connection is presented, the stitching result is better than that of G^1continuity, but the condition of connection is weaker than that of curvature connection. The method is of some value to the computer aided geometric design.

The condition for Gaussian curvature connection of Bézier surfaces

For the Bézier surfaces $r(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{n} b_{i,j} B_{i,n}(u) B_{j,n}(v)$, we know that

$$r_{uu} = n(n-1)\sum_{i=0}^{n-2}\sum_{j=0}^{n} \Delta^{2,0} b_{i,n} B_{i,n-2}(u) B_{j,n}(v),\ r_u = n\sum_{i=0}^{n-1}\sum_{j=0}^{n} \Delta^{1,0} b_{i,j} B_{i,n-1}(u) B_{j,n}(v),$$

$$r_v = n\sum_{i=0}^{n}\sum_{j=0}^{n-1} \Delta^{0,1} b_{i,j} B_{i,n}(u) B_{j,n-1}(v).$$

Let

$$r^{k+1}(\bar{u},\bar{v})=\sum_{i=0}^{n}\sum_{j=0}^{n}a_{i,n}^{k+1}B_{i,n}(\bar{u})B_{j,n}(\bar{v}),\ \ r^{k}(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{n}a_{i,n}^{k}B_{i,n}(u)B_{j,n}(v)\,.$$

Where $a_{i,n}^{k}$ and $a_{i,n}^{k+1}$ $(i=1,2,\ldots,n)$ are control points of $r^{k}(u,v)$ and $r^{k+1}(\bar{u},\bar{v})$ respectively.

If $r^{k+1}(\bar{u},0)=r^{k}(u,1)$, $0\le\bar{u}=u\le1$, we know that $r^{k+1}(\bar{u},\bar{v})$ and $r^{k}(u,v)$ are connected along their common boundary curve, therefore, we have $a_{i,o}^{k+1}=a_{i,n}^{k}$, $i=0,1,\ldots,n$.

If the two surfaces connected with G^1continuity along common boundary $r^{k+1}(\bar{u},0)=r^{k}(u,1)$, we have the follwing conditions[7]

$$r^{k+1}(u,0)=r^{k}(u,1)\,,\ \ r_{\bar{v}}^{k+1}(u,0)=ar_{v}^{k}(u,1)+[b(1-u)+gu]r_{u}^{k}(u,1)=ar_{v}^{k}(u,1)+g(u)r_{u}^{k}(u,1)\,. \tag{1}$$

Conditions in (1) can be written as follows

$$a_{i,o}^{k+1}=a_{i,n}^{k}\,,\ \ \Delta^{0,1}a_{i,0}^{k+1}=a\Delta^{0,1}a_{i,3}^{k}+b\,\frac{n-i}{n}\Delta^{1,0}a_{i,n}^{k}+g\,\frac{i}{n}\Delta^{1,0}a_{i-1,n}^{i}\,,i=0,1,\ldots,n\,.$$

The definition of Gaussian curvature is

$$k=\frac{LN-M^{2}}{EG-F^{2}}\,. \tag{2}$$

Let

$$\bar{E}=r_{\bar{u}}^{k+1}\cdot r_{\bar{u}}^{k+1},\bar{F}=r_{\bar{u}}^{k+1}\cdot r_{\bar{v}}^{k+1},\bar{G}=r_{\bar{v}}^{k+1}\cdot r_{\bar{v}}^{k+1},\bar{L}=\vec{n}\cdot r_{\bar{u}\bar{u}}^{k+1},\bar{M}=\vec{n}\cdot r_{\bar{u}\bar{v}}^{k+1},\bar{N}=\vec{n}\cdot r_{\bar{v}\bar{v}}^{k+1},$$

$$E=r_{u}^{k}\cdot r_{u}^{k},F=r_{u}^{k}\cdot r_{v}^{k},G=r_{v}^{k}\cdot r_{v}^{k},L=\vec{n}\cdot r_{uu}^{k},M=\vec{n}\cdot r_{uv}^{k},N=\vec{n}\cdot r_{vv}^{k}\,.$$

From (1), we get

$$\bar{E}=E\,,\ \bar{G}=[ar_{v}^{k}(u,1)+g(u)r_{u}^{k}(u,1)]^{2}=a^{2}G+2agF+g^{2}E\,,\bar{F}=aF+gE\,.$$

Then we have

$$\bar{E}\bar{G}-\bar{F}^{2}=a^{2}EG+2agFE+g^{2}E^{2}-(aF+gE)^{2}=a^{2}(EG-F^{2})\,. \tag{3}$$

Since $r_{\bar{u}\bar{u}}^{k+1}(u,0)=r_{uu}^{k}(u,1)$, we have $r_{\bar{v}\bar{u}}^{k+1}(u,0)=ar_{vu}^{k}(u,1)+g'(u)r_{u}^{k}(u,1)+g(u)r_{uu}^{k}(u,1)$, and we konw that $\bar{L}=L\,,\bar{M}=aM+g(u)L$.

We get

$$\bar{L}\bar{N}-\bar{M}^{2}=L\bar{N}-[aM+g(u)L]^{2}=L\bar{N}-a^{2}M^{2}-2agML-g^{2}L^{2}\,. \tag{4}$$

From (2),(3) and (4), if the two surfaces are connected with Gaussian curvature continuity along their common boundary $r^{k+1}(u,0)=r^{k}(u,1)$, we have the following equation

$$\bar{L}\bar{N}-\bar{M}^{2}=a^{2}(LN-M^{2})\,. \tag{5}$$

From (4) and (5), we find $L[\bar{N}-(a^{2}N+2agM+g^{2}L)]=0$, then we have

$\bar{N}-(a^{2}N+2agM+g^{2}L)=0$, or $L=0$.

If $\bar{N}-(a^{2}N+2agM+g^{2}L)=0$, we obtain

$$\vec{n}\cdot[r_{\bar{v}\bar{v}}^{k+1}(u,0)-(a^{2}r_{uu}^{k}(u,1)+2agr_{uv}^{k}(u,1)+g^{2}r_{vv}^{k}(u,1))]=0\,,$$

this means that $r^{k+1}_{\bar{v}\bar{v}}(u,0)-(a^2r^k_{uu}(u,1)+2agr^k_{uv}(u,1)+g^2r^k_{vv}(u,1)$, $r^k_u(u,1)$ and $r^k_v(u,1)$ are on the same plane, We have

$$r^{k+1}_{\bar{v}\bar{v}}(u,0)=(a^2r^k_{uu}(u,1)+2agr^k_{uv}(u,1)+g^2r^k_{vv}(u,1))+br^k_u(u,1)+cr^k_v(u,1).$$

This indicates that the surfaces are connected with curvature continuity.

If $L=0$, we find $\vec{n}\cdot r^{k+1}_{\bar{u}\bar{u}}(u,0)=0$, it means that vectors $r^{k+1}_{\bar{u}\bar{u}}(u,0)$, $r^k_u(u,1)$ and $r^k_v(u,1)$ are on the same plane, we can write

$$(a_1(1-u)^2+a_2u(1-u)+a_3u^2)r^{k+1}_{\bar{u}\bar{u}}(u,0)=a_4r^k_v(u,1)+(a_5(1-u)+a_6u)r^k_u(u,1). \tag{6}$$

Accroding to the definition of Bernstein polynormials, we get

$$(1-u)^2B_{i,n-2}(u)=\frac{(n-1-i)(n-i)}{n(n-1)}B_{i,n}(u),\ (1-u)uB_{i,n-2}(u)=\frac{(i+1)(n-1-i)}{n(n-1)}B_{i+1,n}(u),$$
$$u^2B_{i,n-2}(u)=\frac{(i+1)(i+2)}{n(n-1)}B_{i+2,n}(u),\ (1-u)B_{i,n-1}(u)=\frac{n-i}{n}B_{i,n}(u),\ uB_{i,n-1}(u)=\frac{1+i}{n}B_{1+i,n}(u),\ i=1,2,\ldots,n. \tag{7}$$

By using formulas in(7) , equation (6) becomes

$$\sum_{i=0}^{n-2}[a_1\frac{(n-i)(n-1-i)}{n}\Delta^{2,0}a^{k+1}_{i,0}+a_2\frac{i(n-i)}{n}\Delta^{2,0}a^{k+1}_{i-1.0}+a_3\frac{i(i-1)}{n}\Delta^{2,0}a^{k+1}_{i-2,0}]B_{i,n}(u)$$
$$=\sum_{i=0}^{n}[a_4\Delta^{0,1}a^k_{i,n-1}+a_5\frac{n-i}{n}\Delta^{1,0}a^k_{i,n}+a_6\frac{i}{n}\Delta^{1,0}a^k_{i-1}]B_{i,n}(u),\ i=1,2,\ldots,n.$$

Therefore, the equation (6) can also be written as

$$a_1\frac{(n-i)(n-1-i)}{n}\Delta^{2,0}a^{k+1}_{i,0}+a_2\frac{i(n-i)}{n}\Delta^{2,0}a^{k+1}_{i-1.0}+a_3\frac{i(i-1)}{n}\Delta^{2,0}a^{k+1}_{i-2,0}$$
$$=a_4\Delta^{0,1}a^k_{i,n-1}+a_5\frac{n-i}{n}\Delta^{1,0}a^k_{i,n}+a_6\frac{i}{n}\Delta^{1,0}a^k_{i-1,n},\ i=1,2,\ldots,n.$$

Then the coditions for Gausian curvature connction can be written in the form

$$\begin{cases} r^{k+1}(u,0)=r^k(u,1) \\ r^{k+1}_v(u,0)=g(u)r^k_u(u,1)+h(u)r^k_v(u,1) \\ f(u)r^{k+1}_{\bar{u}\bar{u}}(u,0)=s(u)r^k_v(u,1)+l(u)r^k_u(u,1) \end{cases}.$$

Where $h(u)=a$, $g(u)=b(1-u)+gu$, $f(u)=a_1(1-u)^2+a_2u(1-u)+a_3u^2$, $s(u)=a$, $l(u)=a_5(1-u)+a_6u$ and $a,b,g,a_i\,(i=1,2,\ldots,6)$ are all constant and $a>0$.

Acoording to the difference formulas, the conditions for Gausian curvature connctions can be written as follows

(i) $a^{k+1}_{i,o}=a^k_{i,n}$, $i=0,1,2,\ldots,n$.

(ii) $\Delta^{0,1}a^{k+1}_{i,0}=a\Delta^{0,1}a^k_{i,n-1}+b\frac{n-i}{n}\Delta^{1,0}a^k_{i,n}+g\frac{i}{n}\Delta^{1,0}a^i_{i-1,n}$, $i=0,1,2,\ldots,n$.

(iii) $a_1\frac{(n-i)(n-1-i)}{n}\Delta^{2,0}a^{k+1}_{i,0}+a_2\frac{i(n-i)}{n}\Delta^{2,0}a^{k+1}_{i-1.0}+a_3\frac{i(i-1)}{n}\Delta^{2,0}a^{k+1}_{i-2,0}$

$$= a_4 \Delta^{0,1} a_{i,n-1}^k + a_5 \frac{n-i}{n} \Delta^{1,0} a_{i,n}^k + a_6 \frac{i}{n} \Delta^{1,0} a_{i-1,n}^k,\ i = 0,1,2,\ldots,n.$$

From (ii), relation between the points of two adjacent surfaces can be written as follows

$$a_{i,1}^{k+1} = b \frac{n-i}{n} a_{i+1,n}^k + (1 + a - \frac{n-i}{n} b + \frac{i}{n} g) a_{i,n}^k - \frac{i}{n} g a_{i-1,n}^k - a a_{i,n-1}^k,\ i = 0,1,2,\ldots,n. \tag{8}$$

Where $a_{i,1}^{k+1} (i = 0,1,\ldots,n)$ are the second row points of $r^{k+1}(u,v)$, From (iii), the points of $r^k(u,v)$ satisfy the follow equations

$$\begin{aligned} a_4 a_{i,n-1}^k = & -a_3 \frac{(i-1)i}{n} a_{i-2,n}^k + [-a_2 \frac{i(n-i)}{n} + 2a_3 \frac{(i-1)i}{n} - a_6 \frac{i}{n}] a_{i-1,n}^k \\ & - [a_1 \frac{(n-1-i)(n-i)}{n} - 2a_2 \frac{i(n-i)}{n} + a_3 \frac{(i-1)i}{n} - a_4 + a_5 \frac{n-i}{n} - a_6 \frac{i}{n}] a_{i,n}^k \\ & - [-2a_2 \frac{(n-i)(n-1-i)}{n} + a_2 \frac{i(n-i)}{n} - a_5 \frac{n-i}{n}] a_{i+1,n}^k + a_1 \frac{(n-i)(n-i-1)}{n} a_{i+2,n}^k, \end{aligned} \tag{9}$$

$i = 1,2,\ldots,n$.

If $i > n$, we consider that $a_{i,n}^k = \vec{0}$.

The method of Gaussian curvature connection of surfaces

Step 1 input the control points along the common boundary according to the practical problems.

Step 2 determine the coefficients a_i $(i = 1,2,\ldots,6)$) of equations (9) according to actual requirements.

Step 3 find control points $a_{i,n-1}^k$ based on the equations (9).

Step 4 determine the coefficients a, b and g of equations (8).

Step 5 find control points $a_{i,1}^{k+1}$ based on the equations (8).

Step 6 give the rest control points of two surfaces according to the actual requirements.

Then the surfaces are connected with Gaussian curvature continuity along their common boundary.

Conclusion

In this paper we present a new method of connection of Bézier surface, the connection result is better than that with G^1continuity, but the conditions are weaker than that of curvature connection. The method is of some value to the computer aided geometric design.

References

[1] J. Kahmann, Continuity of curvature between adjacent Bézier patches, In: Barnhill R and Boehm W.Surfaces in Computer Aided Geometric Design.The Netherland, 1983, pp65-75.

[2] W. H. Du, F .Schmitt. Computer-Aided Design, 22(1990)556-573.

[3] Q.X. Meng, H.L.Liu, Journal of Computer Aided Design & Computer Graphics, 21(2009)1074-1082. In Chiness.

[4] R.J.Zhang, L.G.Liu, G.J.Wang, Construction of cubic triangular patches with C^1 continuity around a corner, Proceedings of 9th international conference on Computer Aided Design and Computer Graphics 2005 pp73-78.

[5] S.Q.Liu, H.Li, Computer Aided Design 24(1992) 491-503.

[6] A.K.Jonse, Computer Aided Design 20(1988)325-335.

[7] F.Z.Shi, Computer aided geometric design and non-uniform rational B-spline, Beijing, 2001.