

The Higher Order Crack-Tip Fields for Anti-plane Crack in Exponential Functionally Graded Piezoelectric Materials

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Abstract The crack tip fields for anti-plane crack in functionally graded piezoelectric materials (FGPMs) under mechanical and electrical loadings are investigated. The elastic stiffness, piezoelectric parameter and dielectric permittivity of FGPMs are assumed to be exponential function of y perpendicular to the crack with different gradient parameters, respectively. By using the eigen-expansion method, the higher order crack tip stress and electric displacement fields for FGPMs are obtained. The analytic expressions of the stress intensity factors and the electric displacement intensity factors are derived.

Introduction

In recent years, piezoceramics have been widely studied and utilized in numerous applications, e.g., as displacement transducers, sensors and actuators. The mechanical reliability and durability of these materials has become increasingly important. The disadvantage of those materials is that they crack at low temperatures and creep at high temperatures. Therefore fracture of piezoelectric materials have received much attention. The development of functionally graded materials (FGMs) has demonstrated that they have the potential to reduce the stress concentration and to increase the fracture toughness. FGMs can be extended to piezoelectric materials to improve its reliability. Most of literatures assumed that all material properties of FGPMs are exponential functions of coordinates with same gradient parameters. Singh[1] studied the problem of an antiplane crack situated in the interface of two bonded dissimilar graded piezoelectric half-spaces under the permeable crack assumption. Wang[2] investigated the dynamic response of a center-situated crack perpendicular to the edges of the piezoelectric strip subjected to anti-plane mechanical and electrical impacts. Han[3] calculated the plane electro-elastic fields in piezoelectric materials with multiple cracks. Different from previous analyses, the elastic stiffness, piezoelectric parameter and dielectric permittivity of FGPMs are assumed to be exponential function of y perpendicular to the crack with different gradient parameters, respectively. In this paper, we extend the Williams' solution to fracture problem of FGPMs and the stress and electric displacement high order fields are obtained.

Basic equations

Consider a crack in a functionally graded piezoelectric materials under anti-plane shear tractions and in-plane electric displacements, as shown in Fig.1. The FGPM is poled in the z direction and isotropic in the xoy plane. The present work employs exponential function to describe the continuous variations of material properties,

$$c_{44} = c_{440} e^{b_1 r \sin q} \quad e_{15} = e_{150} e^{b_2 r \sin q} \quad e_{11} = e_{110} e^{b_3 r \sin q} \quad (1)$$

where c_{440} is the shear modulus, e_{150} is the piezoelectric coefficient, e_{110} is the dielectric parameter at $x = 0$.

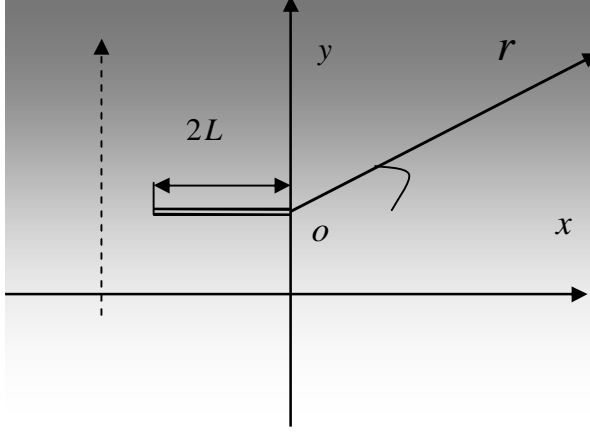


Fig.1 Anti-plane crack in FGPMs

The governing equations can be written as

$$\begin{cases} c_{44} \nabla^2 w + \frac{\partial c_{44}}{\partial r} \frac{\partial w}{\partial r} + \frac{\partial c_{44}}{r^2 \partial q} \frac{\partial w}{\partial q} + e_{15} \nabla^2 f + \frac{\partial e_{15}}{\partial r} \frac{\partial f}{\partial r} + \frac{\partial e_{15}}{r^2 \partial q} \frac{\partial f}{\partial q} = 0 \\ e_{15} \nabla^2 w + \frac{\partial e_{15}}{\partial r} \frac{\partial w}{\partial r} + \frac{\partial e_{15}}{r^2 \partial q} \frac{\partial w}{\partial q} - e_{11} \nabla^2 f - \frac{\partial e_{11}}{\partial r} \frac{\partial f}{\partial r} - \frac{\partial e_{11}}{r^2 \partial q} \frac{\partial f}{\partial q} = 0 \end{cases} \quad (1)$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial q^2}$ is the two-dimensional Laplace operator.

The Higher Order Crack-Tip Field

The displacement component w and the electric potential f can be expanded as follows[4]

$$w = \sum_{i=1}^{\infty} r^{\frac{i}{2}} w_i(q), \quad f = \sum_{i=1}^{\infty} r^{\frac{i}{2}} f_i(q) \quad (2)$$

where, $w_i(q)$ and $f_i(q)$ are eigen-functions.

Substitute Eq. (3) into Eq.(2). According to the linear independence of $r^{-3/2}$, r^{-1} , $r^{-1/2}$, ..., $r^{i/2-2}$, ..., the system of ordinary differential equations are obtained. In the case of electrically impermeable crack, the crack surfaces are free of electric charges and the electric displacement inside the crack is zero. As the crack surface is free, the boundary conditions are

$$S_{zy} \Big|_{q=\pm p} = 0, \quad D_y \Big|_{q=\pm p} = 0 \quad (3)$$

Solving the system of ordinary differential equations, we can obtain the results

$$\begin{cases} w_1(q) = B_{11} \sin \frac{q}{2} \\ f_1(q) = B_{12} \sin \frac{q}{2} \end{cases} \quad (4)$$

$$\begin{cases} w_2(q) = B_{21} \cos q \\ f_2(q) = B_{22} \cos q \end{cases} \quad (5)$$

$$\begin{cases} w_3(q) = B_{31} \sin \frac{3q}{2} + \left[-\frac{e_{150}^2 b_2 + c_{440} e_{110} b_1}{12(c_{440} e_{110} + e_{150}^2)} B_{11} + \frac{e_{150} e_{110} (b_3 - b_2)}{12(c_{440} e_{110} + e_{150}^2)} B_{12} \right] (3 \cos \frac{q}{2} + \cos \frac{3q}{2}) \\ f_3(q) = B_{32} \sin \frac{3q}{2} + \left[-\frac{c_{440} e_{150} (b_2 - b_1)}{12(c_{440} e_{110} + e_{150}^2)} B_{11} - \frac{c_{440} e_{110} b_3 + e_{150}^2 b_2}{12(c_{440} e_{110} + e_{150}^2)} B_{12} \right] (3 \cos \frac{q}{2} + \cos \frac{3q}{2}) \end{cases} \quad (6)$$

$$\begin{cases} w_4(q) = B_{41} \cos 2q \\ f_4(q) = B_{42} \cos 2q \end{cases} \quad (7)$$

$$\begin{aligned} w_5(q) = & B_{51} \sin \frac{5q}{2} + \frac{1}{48(e_{150}^2 + e_{110} c_{440})^2} \{ [(e_{150}^4 - 5c_{440} e_{150}^2 e_{110}) b_2^2 + (c_{440}^2 e_{110}^2 - \\ & 2c_{440} e_{150}^2 e_{110}) b_1^2 + 3c_{440} e_{150}^2 e_{110} (3b_1 b_2 + b_2 b_3 - b_1 b_3)] B_{11} + [(2e_{150}^3 e_{110} - \\ & c_{440} e_{150} e_{110}^2)(2b_2^2 + b_3^2) - 6e_{150}^3 e_{110} b_2 b_3 + 3c_{440} e_{150} e_{110}^2 (b_1 b_2 + b_2 b_3 - \\ & b_1 b_3)] B_{12} \} \sin \frac{q}{2} + \frac{1}{20(e_{150}^2 + e_{110} c_{440})^2} \{ [(c_{440} e_{150}^2 e_{110} + c_{440}^2 e_{110}^2) b_1 + \\ & (c_{440} e_{150}^2 e_{110} + e_{150}^4) b_2] B_{31} + (e_{150}^3 e_{110} + c_{440} e_{150} e_{110}^2)(b_2 - b_3) B_{32} \} (\cos \frac{5q}{2} - \\ & 5 \cos \frac{q}{2}) + \frac{1}{32(e_{150}^2 + e_{110} c_{440})^2} \sin \frac{3q}{2} \{ [(c_{440}^2 e_{110}^2 - 2c_{440} e_{150}^2 e_{110}) b_1^2 + \\ & b_2^2 (e_{150}^4 - 5c_{440} e_{150}^2 e_{110}) + 3c_{440} e_{150}^2 e_{110} (3b_1 b_2 + b_2 b_3 - b_1 b_3)] B_{11} + \\ & [(2e_{150}^3 e_{110} - c_{440} e_{150} e_{110}^2)(2b_2^2 + b_3^2) - 6e_{150}^3 e_{110} b_2 b_3 + 3c_{440} e_{150} e_{110}^2 \cdot \\ & (b_1 b_2 + b_2 b_3 - b_1 b_3)] B_{12} \} \\ f_5(q) = & B_{52} \sin \frac{5q}{2} + \frac{1}{48(e_{150}^2 + e_{110} c_{440})^2} \{ [(c_{440}^2 e_{150} e_{110} - 2c_{440} e_{150}^3)(b_1^2 + \\ & 2b_2^2) + 3c_{440} e_{150} e_{110} (b_1 b_3 - b_2 b_3 - b_1 b_2) + 6c_{440} e_{150}^3 b_1 b_2] B_{11} + \\ & [(c_{440}^2 e_{110}^2 - 2c_{440} e_{150}^2 e_{110}) b_3^2 + (e_{150}^4 - 5c_{440} e_{150}^2 e_{110}) b_2^2 + 3c_{440} e_{150}^2 e_{110} \cdot \\ & (b_1 b_2 + 3b_2 b_3 - 3b_1 b_3)] B_{12} \} \sin \frac{q}{2} + \frac{1}{20(e_{150}^2 + e_{110} c_{440})^2} \{ (c_{440}^2 e_{150} e_{110} + \\ & c_{440} e_{150}^3)(b_2 - b_1) B_{31} + [(c_{440} e_{150}^2 e_{110} - c_{440}^2 e_{110}^2) b_3 - (c_{440} e_{150}^2 e_{110} + e_{150}^4) \cdot \\ & b_2] B_{32} \} (\cos \frac{5q}{2} - 5 \cos \frac{q}{2}) + \frac{1}{32(e_{150}^2 + e_{110} c_{440})^2} \{ [(c_{440}^2 e_{150} e_{110} - \\ & 2c_{440} e_{150}^3)(b_1^2 + 2b_2^2) + 6c_{440} e_{150}^3 b_1 b_2 + 3c_{440} e_{150} e_{110} (b_1 b_3 - b_2 b_3 - \\ & b_1 b_2)] B_{11} + [(c_{440}^2 e_{110}^2 - c_{440} e_{150}^2 e_{110}) b_3^2 + (e_{150}^4 - 5c_{440} e_{150}^2 e_{110}) b_2^2 + \\ & 3c_{440} e_{150}^2 e_{110} (b_1 b_2 + 3b_2 b_3 - b_1 b_3) B_{12}] \} \sin \frac{3q}{2} \end{aligned} \quad (8)$$

$$\begin{cases} w_6(q) = B_{61} \cos 3q + \frac{3 \sin q - \sin 3q}{12(e_{150}^2 + c_{440} e_{110})} [(c_{440} e_{110} b_1 + e_{150}^2 b_2) B_{41} + e_{150} e_{110} (b_2 - b_3) B_{42}] \\ f_6(q) = B_{62} \cos 3q + \frac{3 \sin q - \sin 3q}{12(e_{150}^2 + c_{440} e_{110})} [c_{440} e_{150} (b_1 - b_2) B_{41} + (c_{440} e_{110} b_3 + e_{150}^2 b_2) B_{42}] \end{cases} \quad (9)$$

where $A_{ij} = \{A_{ij} \ B_{ij}\}$ are the undetermined coefficients.

Substituting Eq. (5)-(10) into Eq.(3), the displacement component w and the electric potential f are obtained.

Then, the stress and the electric displacement components can be obtained

$$\begin{cases} t_{xz} = c_{44} w_{,x} + e_{15} f_{,x} \\ t_{yz} = c_{44} w_{,y} + e_{15} f_{,y} \\ D_x = e_{15} w_{,x} - e_{11} f_{,x} \\ D_y = e_{15} w_{,y} - e_{11} f_{,y} \end{cases} \quad (10)$$

The mode III stress intensity factor (SIF) and electric displacement intensity factor (EDIF) of the crack tip are defined as

$$\begin{cases} K^T = \lim_{r \rightarrow 0} \sqrt{2\pi r} S_{yz}(r, 0) = \frac{\sqrt{2\pi}}{2} (c_{440} B_{11} - e_{150} B_{12}) \\ K^D = \lim_{r \rightarrow 0} \sqrt{2\pi r} D_y(r, 0) = \frac{\sqrt{2\pi}}{2} (e_{150} B_{11} + e_{110} B_{12}) \end{cases} \quad (11)$$

Conclusions

The higher order displacement, electric potential, stress and electric displacement fields for exponential FGPMs III-mode crack are obtained in this paper. The results show that the first two items of the higher order crack-tip fields of FGPMs have the same mathematical forms as ones of homogeneous piezoelectric materials. The gradient parameters appear in the third and higher order fields. It is clear that gradient parameters strongly affect the higher order items. Due to coupling effect of piezoelectric material, the stress intensity factor and electric displacement intensity factor are dependent on both displacement component and the electric potential.

Acknowledgments

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