Higher Order Crack Tip Fields for Physical Weak-Discontinuous Crack of Linear FGMs Plate with Reissner's Effect

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Abstract The higher order crack tip fields for an interfacial crack between homogeneous material and linear functionally graded materials (FGMs) plates is studied. The governing equations are derived for weak-discontinuous problems of FGMs plates considering transverse shear deformation . The higher order crack tip fields of homogeneous materials and FGMs regions are obtained by the eigen-expansion method, respectively. Finally, the whole crack tip high order fields are assembled and given. The crack tip fields obtained also have the same nature of eigen-function as Williams' solution.

Introduction

The Reissner plate model is frequently used by engineers in connection with plate and shell problems of small to moderate thickness. Murthy et al. studied symmetrical I-mode crack of Reissner's homogeneous plate and obtained the general term of expansion equation of crack-tip strain field [1]. Liu studied the crack-tip field of Reissner's homogeneous plate with asymptotic expansion method and got the first several items of the asymptotic expansion solutions [2,3]. Qian et al. studied the stress and strain fields at the tip of notch in Reissner's plate, the eigen-functions of notch is derived, and eigenvalues of different notches with different angles were calculated by Muller iteration method [4]. Xu extended the expanding form of stresses and deformation near a crack tip to a general case, and it was shown that the Reissner plate eigen problem was equivalent to a combination form of a plane problem and an anti-plane problem [5]. In the engineering applications, FGMs mainly appear in the coating and interface layer, forming lots of interface structures. Due to the feature of production technology, there are a number of interface defects in structures. Furthermore, due to differences in material performance across interfaces (i.e. the differentials of material parameters at interfaces are different or discontinuous and the material functions are continuous at interfaces, which is defined as the physical weak-discontinuous), the interfacial fracture becomes the main form of structure failure. Consequently, the study on weak-discontinuous fracture of the FGM plate is of great significance. In this paper, the physical weak-discontinuous problem of an interfacial crack is considered. As shown in Fig.1, the area below the interface is homogeneous materials region, and the one above the interface is FGMs region. The gradient direction is along y-axis. In this paper, we extend the Williams' solution to physical weak-discontinuous problem of an interfacial crack between homogeneous material and FGMs plates and the higher order fields are obtained.

The Basic Equations

The elastic modulus function form of FGMs is assumed to be

$$E^{(\mathrm{II})} = E^{(\mathrm{I})}(1+g\,y) = E^{(\mathrm{I})}(1+g\,r\sin q) \tag{1}$$

where, $E^{(1)}$ is the elastic modulus of homogeneous material, and $g \geq 0$ are the non-homogeneity parameter. The variation of Poisson's ratios has very insignificant effect on the stress intensity factor of non-homogeneous materials[6]. So, Poisson's ratio is assumed to be the constant m.

Assumed the transverse loading of the plate is to be zero, the governing equations for the homogenous material plate with Reissner's effect are

$$\begin{cases} D^{(k)}[\frac{\partial^2 j_r^{(k)}}{\partial r^2}+\frac{1}{r}\frac{\partial j_r^{(k)}}{\partial r}-\frac{j_r^{(k)}}{r^2}+\frac{1-m}{2r^2}\frac{\partial^2 j_r^{(k)}}{\partial q^2}+\frac{1+m}{2r}\frac{\partial^2 j_q^{(k)}}{\partial r\partial q}-\frac{3-m}{2r^2}\frac{\partial j_q^{(k)}}{\partial q}]+C^{(k)}(\frac{\partial w^{(k)}}{\partial r}-j_r^{(k)})+ \\ \frac{h^3}{12(1-m^2)}[\frac{\partial E^{(k)}}{\partial r}(\frac{\partial j_r^{(k)}}{\partial r}+\frac{m}{r}\frac{\partial j_q^{(k)}}{\partial q}+\frac{m}{r}j_r^{(k)})+\frac{\partial E^{(k)}}{\partial q}\frac{1-m}{2r}(\frac{1}{r}\frac{\partial j_r^{(k)}}{\partial q}+\frac{\partial j_q^{(k)}}{\partial r}-\frac{j_q^{(k)}}{r})]=0 \\ D^{(k)}[\frac{1+m}{2r}\frac{\partial^2 j_r^{(k)}}{\partial r\partial q}+\frac{3-m}{2r^2}\frac{\partial j_r^{(k)}}{\partial q}+\frac{1-m}{2}\frac{\partial^2 j_q^{(k)}}{\partial r^2}+\frac{1-m}{2r}\frac{\partial j_q^{(k)}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 j_q^{(k)}}{\partial q^2}-\frac{1-m}{2r^2}j_q^{(k)}]+C^{(k)}(\frac{1}{r}\frac{\partial w^{(k)}}{\partial q}- \\ j_q^{(k)})+\frac{h^3}{12(1-m^2)}[\frac{\partial E^{(k)}}{\partial r}\frac{1-m}{2}(\frac{1}{r}\frac{\partial j_r^{(k)}}{\partial q}+\frac{\partial j_q^{(k)}}{\partial r}-\frac{j_q^{(k)}}{r})+\frac{1}{r}\frac{\partial E^{(k)}}{\partial q}(\frac{1}{r}\frac{\partial j_q^{(k)}}{\partial q}+\frac{j_r^{(k)}}{r}+m\frac{\partial j_r^{(k)}}{\partial r})]=0 \\ C^{(k)}[\frac{\partial^2 w^{(k)}}{\partial r^2}+\frac{1}{r}\frac{\partial w^{(k)}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w^{(k)}}{\partial q^2}-(\frac{\partial j_r^{(k)}}{\partial r}+\frac{j_r^{(k)}}{r}+\frac{1}{r}\frac{\partial j_q^{(k)}}{\partial q})]+ \\ \frac{5h}{12(1+m)}[\frac{\partial E^{(k)}}{\partial r}(\frac{\partial w^{(k)}}{\partial r}-j_r^{(k)})+\frac{1}{r}\frac{\partial E^{(k)}}{\partial q}(\frac{1}{r}\frac{\partial w^{(k)}}{\partial q}-j_q^{(k)})]=0 \end{cases} \tag{2}$$

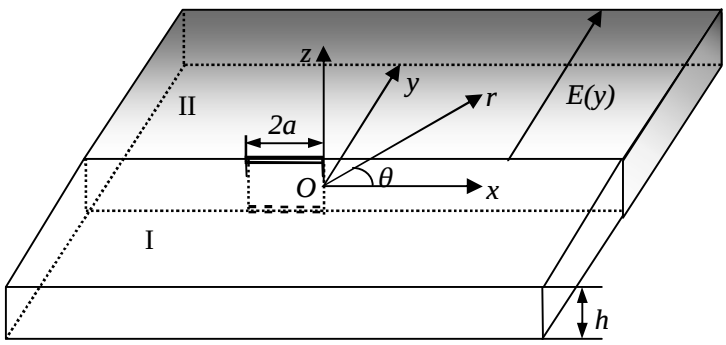

Fig.1The interfacial crack

where $D^{(k)}=\frac{E^{(k)}h^3}{12(1-m^2)}, C^{(k)}=\frac{5E^{(k)}h}{12(1+m)}$ and h is thickness of the plate.

The boundary condition of the crack surface is

$$\begin{cases} \frac{1}{r}\frac{\partial j_q^{(k)}}{\partial q}+\frac{j_r^{(k)}}{r}+m\frac{\partial j_r^{(k)}}{\partial r}=0 \\ \frac{\partial j_r^{(k)}}{\partial q}+\frac{\partial j_q^{(k)}}{\partial r}-\frac{j_q^{(k)}}{r}=0 \qquad q=(-1)^k\pi \\ \frac{1}{r}\frac{\partial w^{(k)}}{\partial q}-j_q^{(k)}=0 \end{cases} \tag{3}$$

Higher Order Crack Tip Fields

The crack tip stress field would be equipped with the same square root singularity as that of homogeneous materials when the material prosperities of different composite materials at the interfaces are continuous[7]. Therefore, the generalized displacements j_r, j_q, w can be expressed as

follows

$$j_r^{(k)}=\sum_{n=1}^{\infty} f_n^{(k)}(q)r^{\frac{n}{2}},\quad j_q^{(k)}=\sum_{n=1}^{\infty} g_n^{(k)}(q)r^{\frac{n}{2}},\quad w^{(k)}=\sum_{n=1}^{\infty} j_n^{(k)}(q)r^{\frac{n}{2}} \tag{4}$$

where, $f_n^{(k)}(q)$、 $g_n^{(k)}(q)$、 $j_n^{(k)}(q)$ are eigen-functions and $k=\mathrm{I},\mathrm{II}$.

Substituting Eq.(4) into Eq.(2) and Eq.(3), the coefficients of $r^{-3/2}, r^{-1},\ldots, r^{n/2-2}$ are linear independent, the each coefficient term must be zero. The obtained equations are solved and the eigen-function are derived as follows

$$\begin{cases} f_1^{(k)} = B_{12}^{(k)}\sin\frac{3q}{2}+B_{11}^{(k)}\cos\frac{3q}{2}+\frac{3m-5}{3(m+1)}B_{12}^{(k)}\sin\frac{q}{2}+\frac{3m-5}{m+1}B_{11}^{(k)}\cos\frac{q}{2} \\ g_1^{(k)} = -\frac{m-7}{m+1}B_{11}^{(k)}\sin\frac{q}{2}-B_{11}^{(k)}\sin\frac{3q}{2}+\frac{m-7}{3(m+1)}B_{12}^{(k)}\cos\frac{q}{2}+B_{12}^{(k)}\cos\frac{3q}{2} \\ j_1^{(k)} = B_{13}^{(k)}\sin\frac{q}{2} \end{cases} \tag{5}$$

$$\begin{cases} f_2^{(k)} = B_{21}^{(k)}\cos 2q+\frac{1-m}{m+1}B_{21}^{(k)} \\ g_2^{(k)} = B_{22}^{(k)}-B_{21}^{(k)}\sin 2q \\ j_2^{(k)} = B_{23}^{(k)}\cos q \end{cases} \tag{6}$$

$$\begin{cases} f_3^{(k)} = B_{31}^{(k)}\cos\frac{5q}{2}+B_{32}^{(k)}\sin\frac{5q}{2}-\frac{5m-3}{5(m+1)}(5B_{31}^{(k)}\cos\frac{q}{2}+B_{32}^{(k)}\sin\frac{q}{2})+\frac{N_m B_{13}^{(k)}}{6(m-1)}(\sin\frac{q}{2}-\sin\frac{5q}{2}-\frac{5m-3}{1+m}\cdot \\ \qquad \cos\frac{q}{2}+\cos\frac{5q}{2})+d_{2k}g[\frac{3}{4}B_{11}^{(k)}\sin\frac{3q}{2}-\frac{1}{4}B_{12}^{(k)}\cos\frac{3q}{2}+\frac{15m+7}{20(m+1)}B_{11}^{(k)}\sin\frac{q}{2}-\frac{27m-13}{36(m+1)}B_{12}^{(k)}\cos\frac{q}{2}] \\ g_3^{(k)} = \frac{m+9}{5(m+1)}(5B_{31}^{(k)}\sin\frac{q}{2}-B_{32}^{(k)}\cos\frac{q}{2})+B_{32}^{(k)}\cos\frac{5q}{2}-B_{31}^{(k)}\sin\frac{5q}{2}-\frac{N_m B_{13}^{(k)}}{6(m-1)}(\sin\frac{5q}{2}+\cos\frac{5q}{2}-\frac{m+9}{1+m}\cdot \\ \qquad \sin\frac{q}{2}-5\cos\frac{q}{2})+d_{2k}g[\frac{5}{12}B_{12}^{(k)}\sin\frac{3q}{2}+\frac{5}{4}B_{11}^{(k)}\cos\frac{3q}{2}+\frac{5(3m+11)}{36(m+1)}B_{12}^{(k)}\sin\frac{q}{2}+\frac{51m+59}{20(m+1)}B_{11}^{(k)}\cos\frac{q}{2}] \\ j_3^{(k)} = B_{33}^{(k)}\sin\frac{3q}{2}+\frac{2(m-1)}{3(m+1)}(3B_{11}^{(k)}\cos\frac{q}{2}+B_{12}^{(k)}\sin\frac{q}{2})+\frac{2(m+7)}{3(m+1)}B_{11}^{(k)}\cos\frac{3q}{2}-\frac{d_{2k}g B_{13}^{(k)}}{12}(\cos\frac{3q}{2}+3\cos\frac{q}{2}) \end{cases} \tag{7}$$

$$\begin{cases} f_4^{(k)} = B_{41}^{(k)}\cos 3q-\frac{3m-1}{3(m+1)}B_{41}^{(k)}\cos q+B_{42}^{(k)}\sin 3q-\frac{3m-1}{m+1}B_{42}^{(k)}\sin q+\frac{B_{23}^{(k)}N_m}{3(m^2-1)}\cos q \\ g_4^{(k)} = B_{42}^{(k)}\cos 3q-\frac{m+5}{m+1}B_{42}^{(k)}\cos q-B_{41}^{(k)}\sin 3q+\frac{m+5}{3(m+1)}B_{41}^{(k)}\sin q-\frac{(2m+1)B_{23}^{(k)}N_m}{3(m^2-1)}\sin q \\ j_4^{(k)} = B_{43}^{(k)}\cos 2q+\frac{B_{22}^{(k)}}{2}\sin 2q-\frac{m-1}{2(1+m)}B_{21}^{(k)} \end{cases} \tag{8}$$

where $d_{2k}=\begin{cases}0, & k=1,\\ 1, & k=2,\end{cases}$ and $B_{ij}^{(k)}$ ($i=1\mathbf{L}\,n;\ j=1,2,3$)are undetermined coefficients.

Continuous Conditions

Substituting Eq.(5)-(8) into Eq.(4), the generalized displacement fields in homogenous material and FGMs region are obtained, and the stress fields will be obtained based on the relationship between the generalized displacement and stress. The stress equations are subject to the

following continuous conditions of stress

$$s_q^{(\mathrm{I})}|_{q=0} = s_q^{(\mathrm{II})}|_{q=0}, \quad t_{rq}^{(\mathrm{I})}|_{q=0} = t_{rq}^{(\mathrm{II})}|_{q=0}, \quad t_{zq}^{(\mathrm{I})}|_{q=0} = t_{zq}^{(\mathrm{II})}|_{q=0} \tag{9}$$

The relations between undetermined coefficients can be obtained as

$$\begin{aligned}
&B_{11}^{(1)} = B_{11}^{(2)}, && B_{12}^{(1)} = B_{12}^{(2)}, && B_{13}^{(1)} = B_{13}^{(2)}, \\
&B_{21}^{(1)} = B_{21}^{(2)}, && B_{22}^{(1)} = B_{22}^{(2)}, && B_{23}^{(1)} = B_{23}^{(2)}, \\
&B_{31}^{(1)} = B_{31}^{(2)} + \frac{g}{4} B_{12}^{(2)}, && B_{32}^{(1)} = B_{32}^{(2)} + \frac{17}{12} g B_{11}^{(2)}, && B_{33}^{(1)} = B_{33}^{(2)}, \\
&B_{41}^{(1)} = B_{41}^{(2)}, && B_{42}^{(1)} = B_{42}^{(2)}, && B_{13}^{(1)} = B_{43}^{(2)},
\end{aligned} \tag{10}$$

Substituting Eq.(5)-(8) and Eq.(10) into Eq.(4), the generalized displacement fields in homogenous material and FGMs regions are obtained finally.

Conclusions

The higher order crack tip fields for interfacial cracks of FGMs plates with Reissner's effect are obtained by using the method of eigen-expansion, and the structure of higher order crack tip fields is analyzed. The first two items of crack-tip higher order fields of FGMs plate have the same mathematical forms as ones of homogeneous materials. The non-homogeneous material parameter g first appeared in the third order field and g^2 items in the fifth order one. The crack-tip fields obtained also have the same property of eigen-function as Williams' solution. They can be applied to the fracture parameter calculation and crack-tip stress analysis in different materials, loads and structures, which is of extensive applicability.

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