New Exact Solutions for a Glycolysis Model

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Keywords: glycolysis model, hyperbolic function, group-invariant solution, solitonicic solution.

Abstract. This Essay has studied a glycolysis model and found two new exact solutions for this model, namely the group-invariant solution for the model obtained by applying the Lie group method, and the solitonicic solution for the model obtained by applying the hyperbolic function method.

Introduction

Consider a glycolysis model below:

$$\begin{cases} \dfrac{\partial \mathrm{u}}{\partial \mathrm{t}} = d_1 \Delta u + \delta - ku - uv^2 \\ \dfrac{\partial \mathrm{v}}{\partial \mathrm{t}} = d_2 \Delta v + ku - v + uv^2 \end{cases} \quad (1)$$

Where, u and v represent the concentrations of two chemical substances respectively; δ represents input flow; k is the active rate of enzyme; d_1 and d_2 are the diffusion coefficients; δ , k, d_1 and d_2 are all positive numbers. Essay [1] has conducted a general analysis on this model while Essays [1-4] have studied whether the solutions for the model exist as well as other issues. This essay has accomplished two new exact solutions for Equation (1) respectively by applying the Lie group method and the hyperbolic function method.

Solitonicic solution for the model

Assume:

$$\begin{cases} \mathrm{u}(x, y, t) = \phi(\omega), \\ \mathrm{v}(x, y, t) = \phi(\omega), \omega = x + \lambda t \end{cases}$$

Substitute it into Equation (1):

$$\begin{cases} d_1 u'' \text{-} \lambda u' + \delta - ku - uv^2 = 0 \quad (2) \\ d_2 v'' \text{-} \lambda v' + ku - v + uv^2 = 0 \quad (3) \end{cases}$$

Assume u and v are the polynomials of T^m and T^n respectively (where T is a hyperbolic tangent function). Provided the balance between the highest order derivative term $d_1 u''$ and the nonlinear term uv^2 in Equation (2) and the balance between the highest order derivative term $d_2 u''$ and the nonlinear term uv^2 in Equation (3), we can get: $m + \mathrm{n} + 1 = n + 3,\ \max(2m+1, 2m+1) = m + 3$, and solve it to get: $m = 2, n = 1$. When $m = 2, n = 1$, we can assume that Equation (1) has the following solutions:

$$\begin{cases} \mathrm{u} = a_0 + a_1 T + a_2 T^2, \\ \mathrm{v} = b_0 + b_1 T + \mathrm{b}_2 T^2, \end{cases} \quad (4)$$

Where T is a hyperbolic tangent function. Substitute (4) into (1), combine the similar terms of T , set the coefficients of T variables to zero, and then we get:

$$P_1 = b_1^3\mathrm{k} + \delta d_1 - 3a_0^2 b_1 \text{-} b_1\delta\gamma^2 + 4\mathrm{k}^2\delta + 2\mathrm{k}^3\delta^2 = 0,$$
$$P_2 = b_1^2\mathrm{k}a_0 + 2a\mathrm{k}^2 - 3a_0^2\lambda\mathrm{k}^3 \text{-} a_1\lambda^2 + 2d_1\mathrm{k}^2 - 6a_0^2\lambda^2\gamma^3 \text{-} a_0\mathrm{k}^2\delta^2 = 0,$$
$$P_3 = a_2^2\mathrm{k}a_0 + b_2^3\mathrm{k}^2 - 9a_0^2\delta\mathrm{k}^3 \text{-} b_1\delta^2 = 0,$$
$$P_4 = a_2^2\mathrm{k}a_0 + 3a_0^2\lambda^2\mathrm{k}^2 - 9a_0^2\delta^3 \text{-} d_2^2\delta^2 + 2b_1\mathrm{k}^2 - 8b_1^2\lambda^2\mathrm{k}^3 \text{-} a_1\delta^2 = 0, \quad (5)$$

Let $PS = \{P_1,\ P_2,\ P_3,\ P_4\}$, and by applying Wu's elimination method we can obtain the characteristic series CS as shown below:

$$C_1 = (\text{-}8\lambda + \sqrt{3}\delta a_2 - 5d_1^2\mathrm{k}^2)(a_1\delta - b_0\lambda) = 0,$$
$$C_2 = (\sqrt{3\lambda}\lambda\mathrm{k}^3 + 5k^2 a_1^2\delta + 9b_1\lambda\mathrm{k}^4)(a_0^2 + 8\delta b_1\mathrm{k}^2) = 0,$$
$$C_3 = b_2^3\mathrm{k}^2 - 9a_0^2\delta\mathrm{k}^3 \text{-} b_2^2\delta^2 - 3a_0^2 b_1^2 a_1 + a_0^3 b_1\lambda = 0,$$
$$C_4 = 4\mathrm{k}^2\delta + 2\mathrm{k}^3\delta^2 - 8a_0^2 a_1^2\mathrm{k}^3 \text{-} a_2\lambda^2 + a_1^2 b_2^2\lambda = 0, \quad (6)$$

Solve $CS = 0$, and we can get:

$$a_2 = \frac{8\lambda + 5d_1^2\mathrm{k}^2}{\sqrt{3}\delta}, b_0 = a_1 = a_0 = 0, b_1 = \frac{\sqrt{24\mathrm{d}_1^2 \text{-} 6\delta^2\lambda}}{3}, b_2 = \frac{3d_2^2 \text{-} d_1^2\delta}{2\lambda} \quad (7)$$

Thus, we can obtain the following solitonic solutions for Equation (1):

$$\begin{cases} \mathrm{u}_1 = \dfrac{8\lambda + 5d_1^2\mathrm{k}^2}{\sqrt{3}\delta}\tanh^2(x+\lambda t), \\ \mathrm{v}_1 = \dfrac{\sqrt{24\mathrm{d}_1^2 \text{-} 6\delta^2\lambda}}{3}\tanh(x+\lambda t) + \dfrac{3d_2^2 \text{-} d_1^2\delta}{2\lambda}\tanh^2(x+\lambda t), \end{cases} \quad (8)$$

When $m = 2, n = 1$, we can assume Equation (1) has the following solutions:

$$\begin{cases} \mathrm{u} = a_0 + a_1 T + a_2 T^2, \\ \mathrm{v} = b_0 + b_1 T, \end{cases}$$

Thus, we can obtain the following solitonic solutions for Equation (1):

$$\begin{cases} \mathrm{u}_2 = \dfrac{9d_1^2 \text{-} 5\mathrm{k}^2}{36}\tanh(x+\lambda t) + \dfrac{3\lambda \text{-} d_1^2\delta}{3\lambda}\tanh^2(x+\lambda t), \\ \mathrm{v}_2 = \dfrac{2(k + \sqrt{\mathrm{d}_1^2(\lambda \text{-} 6\delta^2)})}{3}\tanh(x+\lambda t), \end{cases} \quad (9)$$

Group-invariant solution for the model

First determine the infinitesimal generator for Equation (1). To do this, we can assume:

$$X = \xi(x,y,t,u)\frac{\partial}{\partial x} + \tau(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u} + \varphi(x,y,t,u)\frac{\partial}{\partial v}$$

And

$$X^{(2)} = X + \zeta^{\mathrm{t}}\frac{\partial}{\partial u_t} + \varphi^{\mathrm{t}}\frac{\partial}{\partial v_t} + \zeta^{\mathrm{xx}}\frac{\partial}{\partial u_{xx}} + \varphi^{\mathrm{xx}}\frac{\partial}{\partial v_{xx}}$$

Here

$$\zeta^{\alpha}=D_{\alpha}\varphi-u_{x}D_{\alpha}\xi-u_{t}D_{\alpha}\eta,$$
$$\zeta^{\beta_1\ldots\beta_N}=D_{\alpha}\zeta^{\beta_1\ldots\beta_{N-1}}-u_{x}D_{\alpha}\xi-u_{y}D_{\alpha}\tau-u_{x}D_{\alpha}\tau-u_{x}D_{\alpha}\xi-u_{y}D_{\alpha}\tau-u_{x}D_{\alpha}\tau,$$
$$D_{\alpha}=\frac{\partial}{\partial\alpha}+u_{\alpha}\frac{\partial}{\partial u_{\alpha}}+\sum_{\beta_1}u_{\alpha\beta_1}\frac{\partial}{\partial u_{\beta_1}}+\sum_{\beta_1\beta_2}u_{\alpha\beta_1}\frac{\partial}{\partial u_{\beta_1\beta_2}}+\ldots$$
$$\alpha,\beta_1,\ldots,\beta_N\in\{x,t\},N\in\{1,2\},$$

By setting

$$X^{(2)}[d_1u''\text{-}\lambda u'+\delta-ku-uv^2]\big|_2=0,and\quad X^{(2)}[d_2v''\text{-}\lambda v'+ku-v+uv^2]\big|_3=0. \qquad (10)$$

We can obtain the infinitesimal generator for Equation (1) as well as the solutions for Equation (8) by using Maple software.

$$\begin{aligned}&\xi=-xC_1+\sqrt{t}C_3+2\text{tx}C_4\\&\tau=2txC_1+(t^2+x\text{v})C_3+3t^2C_4\\&\eta=(\text{x}^2\text{-vu})C_2+(\frac{1}{2}x^2+\frac{1}{2}\text{v}^2+x\text{v}t)C_4\\&\varphi=\sqrt{2}t\text{v}C_2+\sqrt{\text{v}}C_3+(4\text{t}^2\text{-x})C_4\end{aligned} \qquad (11)$$

Thus, the infinitesimal generators for Equation (1) are shown below:

$$\begin{aligned}&X_1=2t\frac{\partial}{\partial t}-x\frac{\partial}{\partial x},X_2=\sqrt{2}t\text{v}\frac{\partial}{\partial \text{v}}+(\text{x}^2\text{-vu})\frac{\partial}{\partial u},\\&X_3=(t^2+x\text{v})\frac{\partial}{\partial t}+\sqrt{\text{v}}\frac{\partial}{\partial \text{v}}+\text{u}\frac{\partial}{\partial \text{u}},\\&X_4=3t^2\frac{\partial}{\partial t}+2\text{tx}\frac{\partial}{\partial x}+(4\text{t}^2\text{-x})\frac{\partial}{\partial \text{v}}+(\frac{1}{2}x^2+\frac{1}{2}\text{v}^2+x\text{v}t)\frac{\partial}{\partial u}.\end{aligned} \qquad (12)$$

Solve the following initial value problems:

$$\begin{cases}\frac{d}{d}(t^*,x^*,y^*,u^*)=X(t^*,x^*,y^*,u^*)\\(t^*,x^*,y^*,u^*)\big|_{\varepsilon=0}=(t,x,y,u)\end{cases}$$

We can obtain the one-parameter Lie group for (4): $g:(t,x,y,u)\to(t^*,x^*,y^*,u^*)$:

$$\begin{aligned}&g_1:(t,x,\text{v},u)\to(e^{2\varepsilon_1}t,e^{\varepsilon_1}x,\text{v},u)\\&g_2:(t,x,\text{v},u)\to(e^{-\varepsilon_2+\varepsilon_2^2}t,x+2\varepsilon_2t,\text{v},u)\\&g_3:(t,x,\text{v},u)\to(\frac{\varepsilon_3x^2}{1+2\varepsilon_2}t,\frac{\varepsilon_3x^2}{1+2\varepsilon_3}x,v,\text{e}^{-2\varepsilon_3}u)\\&g_4:(t,x,\text{v},u)\to(\text{e}^{2\varepsilon_4^2+\varepsilon_4}t,\frac{1}{\sqrt{2\varepsilon_4}}\text{e}^{\frac{\varepsilon_4x^2}{1+2\varepsilon_4}}x,\text{e}^{2\varepsilon_2}v,\frac{e^{\varepsilon_4}}{u\varepsilon_4+1}u)\end{aligned} \qquad (13)$$

If $f(t,x,\text{v},u)=0$ is the solution for Equation (1), we may obtain new solutions for Equation (1) in the following forms:

$$
\begin{aligned}
&h_1:(t,x,\mathrm{v},u)\to(e^{-2\varepsilon_1}t,e^{-\varepsilon_1}x,\mathrm{v},u),\\
&h_2:(t,x,\mathrm{v},u)\to(e^{-\varepsilon_2 x-\varepsilon_2^2 t}t,x-2\varepsilon_2 t,\mathrm{v},u),\\
&h_3:(t,x,y,u)\to(t,x,\mathrm{v},u)\to(\frac{-\varepsilon_3x^2}{1-2\varepsilon_2}t,\frac{-\varepsilon_3x^2}{1-2\varepsilon_3}x,v,\mathrm{e}^{2\varepsilon_3}u),\\
&h_4:(t,x,\mathrm{v},u)\to(\mathrm{e}^{2\varepsilon_4^2\text{-}\varepsilon_4}t,\frac{1}{\sqrt{2\varepsilon_4}}\mathrm{e}^{\frac{-\varepsilon_4x^2}{1-2\varepsilon_4}}x,\mathrm{e}^{2\varepsilon_2}v,\frac{-e^{\varepsilon_4}}{-u\varepsilon_4+1}u).
\end{aligned}
\tag{14}
$$

By applying (8) (9) (14), we can obtain new exact solutions for Equation (1); e.g. using (8) (14) and (h_4), we can obtain the following new exact solutions for Equation (1):

$$
\begin{cases}
\mathrm{u}_3=\dfrac{8\lambda+5d_1^2\mathrm{k}^2}{\sqrt{3}\delta}\dfrac{-e^{\varepsilon_4}}{-u\varepsilon_4+1}\tanh^2(\dfrac{1}{\sqrt{2\varepsilon_4}}\mathrm{e}^{\frac{-\varepsilon_4x^2}{1-2\varepsilon_4}}x+\lambda\mathrm{e}^{2\varepsilon_4^2\text{-}\varepsilon_4}t),\\
\mathrm{v}_3=\mathrm{e}^{2\varepsilon_2}[\dfrac{\sqrt{24\mathrm{d}_1^2\text{-}6\delta^2\lambda}}{3}\tanh(\dfrac{1}{\sqrt{2\varepsilon_4}}\mathrm{e}^{\frac{-\varepsilon_4x^2}{1-2\varepsilon_4}}x+\lambda\mathrm{e}^{2\varepsilon_4^2\text{-}\varepsilon_4}t)\\
\qquad+\dfrac{3d_2^2\text{-}d_1^2\delta}{2\lambda}\tanh^2(\dfrac{1}{\sqrt{2\varepsilon_4}}\mathrm{e}^{\frac{-\varepsilon_4x^2}{1-2\varepsilon_4}}x+\lambda\mathrm{e}^{2\varepsilon_4^2\text{-}\varepsilon_4}t)],
\end{cases}
\tag{15}
$$

Acknowledgements

This work was financially supported by the Henan Province Natural Science Foundation (15A110028), Henan Province with cutting edge technology based research project(142300410447) and Henan Agricultural University Natural Science Foundation (30300204).

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