Estimation of Unknown Function of a Class of Nonlinear Weakly Singular Integral Inequality

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***Abstract*—In this paper, we investigate a nonlinear weakly singular integral inequality. The upper bounds of the embedded unknown functions are estimated explicitly by the definitions and rules of conformable fractional differential and conformable fractional integration, the techniques of change of variable, and the method of amplification. The derived results can be applied in the study of qualitative properties of solutions of conformable fractional integral equations.**

Keywords-weakly singular integral inequalities; conformable fractional integral; conformable fractional differential; analysis technique; explicit bound

I. INTRODUCTION

It is well known that integral equations are important tools to investigate the rule of natural phenomena. In the study of the qualitative properties of solutions of integral equations, one often deals with certain weakly singular integral inequalities. In 2008, Ma and Pecaric[1] investigated weakly singular integral inequality

$$u^p(t) \le a(t) + b(t)\int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} f(s) u^q(s) ds. \tag{1}$$

In 2014, Zheng [2] discussed the weakly singular integral inequalities of the following form

$$u(t) \le C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s) u(s) ds + \frac{1}{\Gamma(\alpha)}\int_0^A (A-s)^{\alpha-1} g(s) u(s) ds. \tag{2}$$

$$u(t) \le C + \int_0^t h(s) u^p(s) ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s) u^q(s) ds + \int_0^A h(s) u^p(s) ds + \frac{1}{\Gamma(\alpha)}\int_0^A (A-s)^{\alpha-1} g(s) u^q(s) ds. \tag{3}$$

With the development of the theory of differential equations, integral inequalities have been paid much attention by many authors. We refer to the papers [3-10] and the references cited therein.

In this paper, on the basis of [2, 7, 8], we discuss weakly singular integral inequality

$$u(t) \le c + \int_a^t (s-a)^{\alpha-1} f(s) w_1(u(s)) ds + \int_a^t (s-a)^{\alpha-1} g(s) w_1(u(s))[u(s) + \int_a^s (\tau-a)^{\alpha-1} h(\tau) w_2(u(\tau)) d\tau] ds \tag{4}$$

In order to investigate the integral inequality (4), we shall state some basic notations and lemmas, which will be used in the proofs of our main results.

Definition 1. (see [11, 12]) The (left) conformable fractional derivative starting from a of a function $f:[a,\infty) \to R_+$ of order $0 < \alpha \le 1$ is defined by

$$T_\alpha^a(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon}, \tag{5}$$

for all $t > 0$. If f is α-differentiable in some $[a,\infty)$, and $\lim_{t \to a^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to a^+} f^{(\alpha)}(t) \tag{6}$$

Definition 2. (see [11, 12]) The (left) conformable fractional integral starting from a of a function $f:[a,\infty) \to R_+$ of order $0 < \alpha \le 1$ is defined by

$$I_\alpha^a(f)(t)=\int_a^t (s-a)^{\alpha-1} f(s)ds \tag{7}$$

Lemma 1. (see [11, 12]) Let a,b,p,λ,α are real constants with $\alpha\in(0,1]$ and f,g be α -differentiable at a point $t>0$ .Let $h(t)=f(g(t))$ .Then

$$T_\alpha^a(\lambda)=0, \tag{8}$$

$$T_\alpha^a(t^p)=pt^{p-\alpha}, \tag{9}$$

$$T_\alpha^a(af+bg)=aT_\alpha^a(f)+bT_\alpha^a(g), \tag{10}$$

$$T_\alpha^a(fg)=fT_\alpha^a(g)+gT_\alpha^a(f), \tag{11}$$

$$T_\alpha^a(I_\alpha^a(f))(t)=f(t), \tag{12}$$

$$I_\alpha^a(T_\alpha^a(f))(t)=f(t)-f(a), \tag{13}$$

$$T_\alpha^a(h)(t)=T_\alpha^a(f)(g(t))T_\alpha^a(g)(t)g^{\alpha-1}(t). \tag{14}$$

If, in addition, f is differentiable, then

$$T_\alpha^a(f)(t)=(t-a)^{1-\alpha}\frac{df(t)}{dt}. \tag{15}$$

II. MAIN RESULT

Throughout this paper, let $R_+=[0,+\infty)$. Define three functions by w_1,w_2 in (4)

$$W_1(t)=\int_0^t\frac{(s-a)^{\alpha-1}s^{1-\alpha}ds}{w_1(s)},\ t\in R_+,$$
$$W_2(t)=\int_0^t\frac{(s-a)^{\alpha-1}s^{1-\alpha}ds}{W_1^{-1}(s)},\ t\in R_+,$$

$$W_3(t)=\int_0^t\frac{(s-a)^{\alpha-1}w_1(W_1^{-1}(W_2^{-1}(s)))W_1^{-1}(W_2^{-1}(s))ds}{s^{\alpha-1}w_2(W_1^{-1}(W_2^{-1}(s)))}, \tag{6}$$

for all $t>0$.

Theorem 1. Suppose that $f,g,h\in C(R_+,R_+)$, w_1,w_2, $w_2/w_1\in C(R_+,R_+)$ are all nondecreasing and positive functions, and c is a nonnegative constant. If $u(t)$ satisfies (4), then

$$u(t)\le W_1^{-1}\{W_2^{-1}[W_3^{-1}(\Xi(t)]\},\ t\in[0,T_1], \tag{17}$$

where

$$\Xi(t)=W_3\ \{W_2\ [W_1\ (c)+\int_a^t(s-a)^{\alpha-1}f(s)ds]$$
$$+\int_a^t(s-a)^{\alpha-1}g(s)ds\}+\int_a^t(s-a)^{\alpha-1}h(s)ds, \tag{18}$$

and T_1 is the largest number such that

$$\Xi(t)\le W_3(\infty),\ W_3^{-1}(\Xi(T_1))\le W_2(\infty),$$
$$W_2^{-1}(W_3^{-1}(\Xi(T_1)))\le W_1(\infty). \tag{19}$$

Proof. Define a function $z_1(t)$ by the right hand side of the inequality (4), i. e.

$$z_1(t)=c+\int_a^t(s-a)^{\alpha-1}f(s)w_1(u(s))ds$$
$$+\int_a^t(s-a)^{\alpha-1}g(s)w_1(u(s))[u(s)$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^s(\tau-a)^{\alpha-1}h(\tau)w_2(u(\tau))d\tau]ds. \tag{20}$$

We observe that $z_1(t)$ is a positive and nondecreasing function on $[a,\infty)$. From (4) and (20) we have

$$u(t)\le z_1(t),\ t\in[a,\infty), \tag{21}$$

$$z_1(a)=c. \tag{22}$$

Using the define 2 of the (left) conformable fractional integral starting from a of a function and (20)

$$z_1(t) = c + I_\alpha^a(f(t)w_1(u(t))) + I_\alpha^a\{g(t)w_1(u(t))$$
$$\times[u(t) + \int_a^t (\tau-a)^{\alpha-1}h(\tau)w_2(u(\tau))d\tau]\}, t \in R_+. \tag{23}$$

By the lemma 1, we get

$$T_\alpha^a(z_1)(t) = f(t)w_1(u(t)) + g(t)w_1(u(t))[u(t)$$
$$+\int_a^t (\tau-a)^{\alpha-1}h(\tau)w_2(u(\tau))d\tau], t \in R_+. \tag{24}$$

From (21) and (24), we have

$$T_\alpha^a(z_1)(t) \le f(t)w_1(z_1(t)) + g(t)w_1(z_1(t))[z_1(t)$$
$$+\int_a^t (\tau-a)^{\alpha-1}h(\tau)w_2(z_1(\tau))d\tau]\}, t \in R_+. \tag{25}$$

Define a function $z_2(t)$ by

$$z_2(t) = z_1(t) + \int_a^t (\tau-a)^{\alpha-1}h(\tau)w_2(z_1(\tau))d\tau$$
$$= z_1(t) + I_\alpha^a(h(t)w_2(z_1(t))), t \in R_+. \tag{26}$$

Obviously, $z_2(t)$ is a positive and nondecreasing function on $[a,\infty)$. From (22) and (26), we have

$$z_1(t) \le z_2(t), t \in [a,\infty), \tag{27}$$

$$z_2(a) = c. \tag{28}$$

Making a conformable fractional derivative starting from a of the function $z_2(t)$ of order α , by the lemma 1, we obtain

$$T_\alpha^a(z_2)(t) = T_\alpha^a(z_1)(t) + h(t)w_2(z_1(t))$$
$$\le f(t)w_1(z_1(t)) + g(t)w_1(z_1(t))z_2(t) + h(t)w_2(z_1(t))$$
$$\le f(t)w_1(z_2(t)) + g(t)w_1(z_2(t))z_2(t)$$
$$+h(t)w_2(z_2(t)),\ t \in R_+. \tag{29}$$

By the definition of W_1 and the lemma 1, from (29) we obtain

$$T_\alpha^a W_1(z_2(t)) = \frac{z_2^{1-\alpha}(t)}{w_1(z_2(t))}T_\alpha^a(z_2)(t)z_2^{\alpha-1}(t)$$

$$\le f(t) + g(t)z_2(t) + h(t)\frac{w_2(z_2(t))}{w_1(z_2(t))}, t \in R_+. \tag{30}$$

Substituting t with τ in (30), making a conformable fractional integral of order α for (30) with respect to τ from 0 to t and using the lemma 1, we obtain

$$W_1(z_2(t)) \le W_1(z_2(a)) + \int_a^t (s-a)^{\alpha-1}f(s)ds$$
$$+\int_a^t (s-a)^{\alpha-1}g(s)z_2(s)ds$$
$$+\int_a^t (s-a)^{\alpha-1}h(s)\frac{w_2(z_2(s))}{w_1(z_2(s))}ds$$
$$\le W_1(z_2(a)) + \int_a^T (s-a)^{\alpha-1}f(s)ds$$
$$+\int_a^t (s-a)^{\alpha-1}g(s)z_2(s)ds$$
$$+\int_a^t (s-a)^{\alpha-1}h(s)\frac{w_2(z_2(s))}{w_1(z_2(s))}ds, t \in [a,T]. \tag{31}$$

where $T \in [0,T_1]$ is chosen arbitrarily. Let z_3 denote the right hand side of the inequality (31), i. e.

$$z_3(t) = W_1(z_2(a)) + \int_a^T (s-a)^{\alpha-1}f(s)ds$$
$$+\int_a^t (s-a)^{\alpha-1}g(s)z_2(s)ds$$
$$+\int_a^t (s-a)^{\alpha-1}h(s)\frac{w_2(z_2(s))}{w_1(z_2(s))}ds, t \in [a,T]. \tag{32}$$

which is a positive and nondecreasing function on $[a,T]$. From (31) and (32) we have

$$z_2(t) \le W_1^{-1}(z_3(t)), t \in [a,T], \tag{33}$$

$$z_3(a) = W_1(z_2(a)) + \int_a^T (s-a)^{\alpha-1}f(s)ds. \tag{34}$$

From (32) and (33), we have

$$T_\alpha^a(z_3)(t) = g(t)z_2(t) + h(t)\frac{w_2(z_2(t))}{w_1(z_2(t))}$$

$$\le g(t)W_1^{-1}(z_3(t)) + h(t)\frac{w_2(W_1^{-1}(z_3(t)))}{w_1(W_1^{-1}(z_3(t)))}, \tag{35}$$

for all $t \in [a,T]$.

Using the definition of W_2 and the lemma 1, from (35) we get

$$T_\alpha^a(W_2(z_3(t))) = \frac{z_3^{1-\alpha}(t)}{W_1^{-1}(z_3(t))} T_\alpha^a(z_3)(t) z_3^{\alpha-1}(t)$$
$$\le g(t) + h(t)\frac{w_2(W_1^{-1}(z_3(t)))}{w_1(W_1^{-1}(z_3(t)))W_1^{-1}(z_3(t))}, t \in [a,T]. \quad (36)$$

Substituting t with τ in (36), making a fractional integral of order α for (36) with respect to τ from 0 to t and using the lemma 1, we obtain that

$$W_2(z_3(t)) \le W_2(z_3(a)) + \int_a^t (s-a)^{\alpha-1} g(s) ds$$
$$+ \int_a^t (s-a)^{\alpha-1} h(s) \frac{w_2(W_1^{-1}(z_3(s)))}{w_1(W_1^{-1}(z_3(s)))W_1^{-1}(z_3(s))} ds$$
$$\le W_2(z_3(a)) + \int_a^T (s-a)^{\alpha-1} g(s) ds$$
$$+ \int_a^t (s-a)^{\alpha-1} h(s) \frac{w_2(W_1^{-1}(z_3(s)))}{w_1(W_1^{-1}(z_3(s)))W_1^{-1}(z_3(s))} ds, \quad (37)$$

for all $t \in [a,T]$.

Let z_4 denote the right hand side of the inequality (37), i. e.

$$z_4(t) = W_2(z_3(a)) + \int_a^T (s-a)^{\alpha-1} g(s) ds$$
$$+ \int_a^t (s-a)^{\alpha-1} h(s) \frac{w_2(W_1^{-1}(z_3(s)))}{w_1(W_1^{-1}(z_3(s)))W_1^{-1}(z_3(s))} ds, \quad (38)$$

for all $t \in [a,T]$.

From (37) and (38) we have

$$z_3(t) \le W_2^{-1}(z_4(t)), t \in [0,T], \quad (39)$$

$$z_4(a) = W_2(z_3(a)) + \int_a^T (s-a)^{\alpha-1} g(s) ds. \quad (40)$$

From (38) and (39) we have

$$T_\alpha^a(z_4)(t) = h(t)\frac{w_2(W_1^{-1}(z_3(t)))}{w_1(W_1^{-1}(z_3(t)))W_1^{-1}(z_3(t))}$$
$$\le h(t)\frac{w_2(W_1^{-1}(W_2^{-1}(z_4(t))))}{w_1(W_1^{-1}(W_2^{-1}(z_4(t))))W_1^{-1}(W_2^{-1}(z_4(t)))}, \quad (41)$$

for all $t \in [0,T]$.

From (41) we get

$$T_\alpha^a(W_3(z_4(t))) = \frac{w_1(W_1^{-1}(W_2^{-1}(z_4(t))))W_1^{-1}(W_2^{-1}(z_4(t)))}{z_4^{\alpha-1}(t)w_2(W_1^{-1}(W_2^{-1}(z_4(t))))}$$
$$\times T_\alpha^a(z_4(t))z_4^{\alpha-1}(t) \le h(t), t \in [a,T]. \quad (42)$$

From (42) we have

$$W_3(z_4(t)) \le W_3(z_4(a)) + \int_a^t (s-a)^{\alpha-1} h(s) ds, \quad (43)$$

for all $t \in [0,T]$.

From (21), (27), (33) and (39), we get

$$u(t) \le z_1(t) \le z_2(t) \le W_1^{-1}(z_3(t))$$
$$\le W_1^{-1}(W_2^{-1}(z_4(t))), \ t \in [a,T]. \quad (44)$$

From (28), (34), (40), (43) and (44), we have

$$u(t) \le W_1^{-1}\{W_2^{-1}[W_3^{-1}(W_3(W_2(W_1(c)$$
$$+ \int_a^T (s-a)^{\alpha-1} f(s) ds) + \int_a^T (s-a)^{\alpha-1} g(s) ds)$$
$$+ \int_a^t (s-a)^{\alpha-1} h(s) ds)]\}, t \in [0,T]. \quad (45)$$

Because $T \in [0, T_1]$ is chosen arbitrarily, we obtain the required estimation (17). The proof is completed.

III. Summary

In this paper, we investigate a nonlinear weakly singular integral inequality

$$u(t) \le c + \int_a^t (s-a)^{\alpha-1} f(s) w_1(u(s)) ds$$
$$+ \int_a^t (s-a)^{\alpha-1} g(s) w_1(u(s))[u(s)$$
$$+ \int_a^s (\tau-a)^{\alpha-1} h(\tau) w_2(u(\tau)) d\tau] ds.$$

By the definitions and rules of conformable fractional differential and conformable fractional integration, the techniques of change of variable and the method of amplification, we obtain the upper bounds of the embedded unknown functions:

$$u(t) \le W_1^{-1}\{W_2^{-1}[W_3^{-1}(W_3(W_2(W_1(c)$$
$$+\int_a^T (s-a)^{\alpha-1} f(s)ds) + \int_a^T (s-a)^{\alpha-1} g(s)ds)$$
$$+\int_a^t (s-a)^{\alpha-1} h(s)ds)]\}, t \in [0,T].$$

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