

# Image Processing and Analysis Based on Partial Differential Equation Method

Yeyang Jiang<sup>1,a</sup>, Tianfu Chai<sup>2,b</sup>

1 Jiangxi Science & Technology Normal University, Nanchang, China

2 No. 1 Middle School of Wannian, Jiangxi, China

a jyyang1018@126.com, b lijianfang1988@163.com

**Keywords:** Image processing; PDE; Image feature detection

**Abstract.** This paper analyzes the characteristics of image processing method based on nonlinear evolution equations, expounds the advantages, mechanism and theoretical basis of nonlinear evolution equations model in image processing algorithm, and uses the method of partial differential equation for image denoising and edge sharpening (Deblurring), image resolution enhancement, image shaping and image measurement.

## I. INTRODUCTION

Digital image processing is the mathematical processing of converted digital images of simulation image sampling and quantization by use of digital computer technology and other hardware in order to improve the practicability of the image and achieve some expected results.

In information society, digital image processing, no matter in theory or in practice, has a huge potential: the image is an important source for people to obtain information from the objective world; Image information processing is an important means of expanding human vision; Image processing technology has important significance to national industrial, economic and social development.

Along with the development of image processing hardware and software, image processing disciplines increasingly need the intervention and boost of modern mathematical tools. Mathematics provides not only image information representation and coding style and language, but also provides direct foundation and core algorithm for image information processing, processing and utilization. Originally it was from physics and mechanics' nonlinear partial differential equations, which has opened up a new field in image processing and computer vision in recent years. A large number of literature and academic conference based on nonlinear partial differential equation research has received widespread attention and has made great success.

The thought of using partial differential equations in image processing can be traced back to D. Gabor and A.K.Jain's work. However, the field's substantial founding work was due to J.J.Koenderink and A. Pwtkin's independent study: they introduced scale space theory, namely image multi-scale expression, which laid foundation for the application of partial differential equations in image processing.

## II. LINEAR DIFFUSION

A lot of problems in image processing and computer vision applications rely on image quality. However, image degradation happens in the process of formation, transmission and storage. Usually, there are two reasons for this: one is deterministic, which relates to image acquisition mode, imaging generating system defects (for example, incorrect lens adjustment and motion blur) and other factors (for example, atmospheric disturbances); The other one is random, which comes from the noise in the process of signal transmission.

In order to eliminate or reduce the effect of the degrading, structure and actual situation as far as possible approaching the degradation model is very important. However, to identify the random distribution of a given noise in real images is usually not possible. Therefore, in general, after put

forward some assumptions, the degradation model below is formed: assume  $u(x, y): \Omega \subset R^2 \rightarrow R$  is to describe a real scene's original image,  $u_0(x, y)$  is the scene's observe images ( $u(x, y)$ 's degradation), we have

$$u_0(x, y) = h(x, y) * u(x, y) + n(x, y) \quad (1)$$

Here  $h(x, y)$  is the degradation linear operator (system function),  $n(x, y)$  is the additive noise.

Scale space is one family of assignment operator  $\{T_t\}_{t \geq 0}: C_b^\infty(R^2) \rightarrow C_b(R^2)$ ,  $(C_b^\infty(R^2))C_b(R^2)$  are respectively (with arbitrary derivative) bounded continuous function spaces,  $t$  is the scale parameter. The operator acts on the image function  $u_0(x, y)$ , and get different scale images  $u(x, y, t) = (T_t u_0)(x, y)$ .

From the point of view of image processing, scale operator needs to meet the requirements of form. For linear scale operator  $\{T_t, t \geq 0\}$ , if meet the recursive nature, regularity, locality and comparison principle, gray translation, space translation and isometric in-variance, so  $u(x, y, t) = (T_t u_0)(x, y)$  is the solution of the following heat conduction equation:

$$\begin{cases} \frac{\partial u}{\partial t}(x, y, t) = \Delta u(x, y, t), & R^2 \times [0, \infty] \\ u(x, y, 0) = u_0(x, y), & R^2 \end{cases} \quad (2)$$

Expand  $u_0(x, y)$  defined in a rectangular area  $\Omega$  through symmetry and cycle continuation into  $R^2$  area, we find the solution of the equations is the initial image's Gaussian convolution:

$$u(x, y, t) = \int_{R^2} G_{\sqrt{2t}}(x - \eta, y - \xi) u_0(\eta, \xi) d\eta d\xi = (G_{\sqrt{2t}} * u_0)(x, y),$$

Here  $G_\sigma(x, y)$  is the two-dimensional Gaussian kernel function:

$$G_\sigma(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right).$$

Equation (2)'s time parameters has a corresponding relationship with scale parameters of Gaussian kernel function  $\sigma = \sqrt{2t}$ : with the larger scale parameter  $t$ , the mean square deviation of Gaussian convolution grows, and the solution of equation (2) is becoming more and more smooth.

In this way, we will establish the equivalence of Gaussian convolution and heat conduction equation: image makes convolution with Gaussian function with increasing variance, which is equivalent to the solution to heat conduction equation with original image as initial value. Gaussian convolution is the basic operation of a image's linear low-pass filtering. However, Gaussian filtering (heat conduction equation) is isotropic diffusion. At the same time in removing image noise, fuzzy features such as image edges. Improve the filtering technology, and remove noise while preserving important characteristics of information such as edge is always the goal of unremitting efforts to this field.

### III. NONLINEAR DIFFUSION

As a two-dimensional signal, gray image has the character of non-stationary, and the image contains a large number of local characteristics and structure. The linear processing method is difficult to depict these properties, and obtains good processing result; Adaptive nonlinear image processing method is widely used, and shows great superiority.

P. Perona and J. Malik proposed an-isotropic diffusion equation on image edge instead of isotropic diffusion equation (Gaussian smoothing filter):

$$\begin{cases} \frac{\partial u}{\partial t} = \text{div}(g(|\nabla u|)\nabla u) \\ u(0) = u_0 \\ \frac{\partial u}{\partial N} = 0 \end{cases} \quad (3)$$

Here  $\text{div}$  is the divergence operator,  $g(s)$  is a non increased scalar diffusion coefficient:  $g(0) = 1$ ,  $g(s) \rightarrow 0(s \rightarrow \infty)$ . If select  $g(s) = 1$ , then the typical choice of heat conduction

$$g(|\nabla u|) = \frac{1}{1 + \left(\frac{|\nabla u|}{K}\right)^2}$$

equation is

Here,  $K$  is a gradient worshipping value. Equation (3) implements selective diffusion smooth according to image gradient amplitude: for the edge area, the image has large gradient,  $g(s)$  obtains a smaller value, and the equation implements weaker smooth in order to protect the edge character; For flat area, images have smaller gradient,  $g(s)$  has greater value, and strong smooth equations are used to remove image noise.

#### IV. IMAGE RESTORATION

Image restoration is an important research subject in image processing and computer vision. Given an observation image  $f$ , the purpose of image restoration is to reconstructed the best image  $u$  according to the following model :

$$f = Ku + n, \quad (4)$$

In it,  $K$  is a linear operator, representing fuzzy (usually convolution);  $n$  is additive noise (usually a Gaussian noise); And  $u$  and  $f$  are initial images and observed images respectively.

To solve (4) is a morbid inverse problem. Commonly used mathematical method is to add the regularization in energy function to overcome ill-pose. Tikhonov and Arsenin considered the following model:

$$\min_u \int_{\Omega} |Du|^2 + \frac{\lambda}{2} \|Ku - f\|_2^2,$$

where  $\Omega \subset R^N$  is a bounded open neighborhood with Lipschitz boundary,  $Du$  is the derivative of  $u$  distribution. Using this model can reduce the shock caused by high frequency noise, but because it has isotropic smooth properties, leading to bad protection of the image edge and corner. In order to overcome the disadvantages, scholars Rudin and Osher proposed a total variation (TV) based on the variation PDE regularization model:

$$\min_u \int_{\Omega} |Du| + \frac{\lambda}{2} \|Ku - f\|_2^2. \quad (5)$$

This approach classifies image in the bounded variation function space, and uses total variation as a measure to the smoothness of image, so as to get rid of the noise in image at the same time make the jump edges well maintained. Chavent and Kunisch put forward a full bounded variation (i.e., norm) regularized image restoration model

$$\min_u \int_{\Omega} |Du| + \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2 \quad (6)$$

And studied its viscosity solution of the problem.

Contrast model (6) and ROF model (5), it is easy to see that, in model (6) to, a secondary regularization is additionally added to a regular part of the items. Obviously, when  $\alpha = 0$ , model (6) is the standard ROF model. Therefore, this model is promoting ROF model. Additional secondary regularization has two main advantages: first, it provides a mandatory item for the child space founded by kernel functions which belong to gradient operator function; second (and perhaps the most important), it provides a powerful guarantee for the distinguish of bounded variation

containing secondary regularization and non secondary solution structure of bounded variation regularization. We will also further research the regularized image restoration model (6) based on the norm ( $\|u\|_{BV} + \rho \|u\|_2^2$ ), promote split Bregman iterative algorithm to find the numerical solution of (6). Optimization problem (6) can be written as the form below

$$\min_u \|Du\|_1 + \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2.$$

First replace its' variables  $Du \rightarrow d$ , then the following constrained optimization problem can be produced

$$\min_{u,d} \|d\|_1 + \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2, \text{ s.t. } d = Du$$

Replace and directly solve it, we consider its corresponding unconstrained optimization problems

$$\min_{u,d} \|d\|_1 + \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2 + \frac{\gamma}{2} \|d - Du\|_2^2.$$

To solve its split Bregman, iterative algorithm can be described as

$$\begin{aligned} u^{k+1} &= \arg \min_u \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2 + \frac{\gamma}{2} \|Du - d^k + b^k\|_2^2, \\ d^{k+1} &= \arg \min_d \|d\|_1 + \frac{\gamma}{2} \|d - Du^{k+1} - b^k\|_2^2, \end{aligned}$$

And the update formula of  $b^k$

$$b^{k+1} = b^k + (Du^{k+1} - d^{k+1}).$$

Specifically, in a given initial value  $u^0 = 0$  and  $d^0 = b^0 = 0$ , you can get a split Bregman iterative algorithm without restraint

$$\begin{cases} u^{k+1} = \arg \min_u \frac{\alpha}{2} \|u\|_2^2 + \frac{\beta}{2} \|Ku - f\|_2^2 + \frac{\gamma}{2} \|Du - d^k + b^k\|_2^2, \\ d^{k+1} = \arg \min_d \|d\|_1 + \frac{\gamma}{2} \|d - Du^{k+1} - b^k\|_2^2, \\ b^{k+1} = b^k + (Du^{k+1} - d^{k+1}). \end{cases} \quad (7)$$

The above algorithm includes two uncoupled sub-problems. To solve the sub-problems with variable  $u$ , we can get the following optimization conditions

$$0 = \alpha u^{k+1} + \beta K^T (Ku^{k+1} - f) + \gamma D^T (Du^{k+1} - d^k + b^k),$$

This means that

$$u^{k+1} = (\alpha I + \beta K^T K - \gamma \Delta)^{-1} (\beta K^T f + \lambda D^T (d^k - b^k)), \quad (8)$$

In it,  $D^T = -div$  is the ad-joint operator of  $D$ , and meet  $\Delta = -D^T D$ . Consider the system (8) is a strictly diagonally dominant, therefore sub-problems of  $u$  can be solved through Gauss-Seidel iterative algorithm quickly.

To solve the sub-problems of  $d$ , use promotion contraction principle, namely

$$d^{k+1} = shrink(Du^{k+1} + b^k, \frac{1}{\gamma}) = \max(\|Du^{k+1} + b^k\|_2 - \frac{1}{\gamma}, 0) \frac{Du^{k+1} + b^k}{\|Du^{k+1} + b^k\|_2}.$$

Because the law only includes scalar and matrix multiplication operation, which makes numerical calculation easy, in turn, a fast iterative algorithm is produced to ensure the sub-problems of  $d$  can be effectively solved.

This alternating minimization algorithm actually contains both inside and outside two iterative loops. And for each given value  $\lambda$ , there is a corresponding external iteration. As a result of optional  $\lambda$  value, we can choose a best  $\lambda$ , make the system (7) reach steady state solution in the shortest possible time. In addition, according to literature [15], if internal number of iterations in the algorithm is too big, the convergence cannot be speed up, and can damage the accuracy of the internal iteration. In view of this, we usually assume that the internal number of iterations is 1. The biggest advantage of this algorithm is fast convergent speed, and moreover it is easy for program implementation.

## V. RESULTS ANALYSIS

Partial differential equation method is applied in image processing with comprehensive utilization of modern mathematical tools such as partial differential equations, differential geometry, vector analysis and field theory, computational fluid dynamics and bounded variation space and viscosity solution theory. Its basic idea is to evolve an image, a curve, or a surface in a partial differential equation model evolution and get the desired results by numerical solving the partial differential equation. In essence, through effective numerical discrete, partial differential equation is transformed into a nonlinear local iterative filter.

The successful application of Nonlinear partial Differential Evolving Equations mainly due to this method's three basic characteristics: "to be local", "to be iterative" and "to be feature dependant (adaptive)". "Nonlinear" means that the equation parameters and the coefficient are dependent on the image (equation solution) and its characteristics; "Partial Differential" means local processing algorithm; and "Evolving" means that the iterative process of numerical implementation algorithm. Therefore, the literal meaning of "Nonlinear Partial Differential Evolving Development Equations" has revealed the characteristics and nature of this method.

## ACKNOWLEDGMENT

This work is supported by PhD research startup foundation of Jiangxi Science and Technology Normal University the National Natural Science Foundation of China (No. 11171222).

## REFERENCES

- [1] Ruan Qiuqi. Digital image processing (second edition). Beijing: Electronic Industry Press, 2007.
- [2] Chan T, Shen J. Image Processing and Analysis: Variational, PDE, Wavelet, and Stochastic Methods, Philadelphia: the Society for Industrial and Applied Mathematics, 2005.
- [3] Aubert G, Korn Probst: Partial Differential Equations and the Calculus of Variations (Second Edition). New York: Springer-Verlag, 2006.
- [4] Sapiro G.: Geometric Partial Differential Equations and Image Analysis. Cambridge: Cambridge University Press, 2001.
- [5] Gabor D. Information theory in electron microscopy. Laboratory investigation, 1965, 14:801-807.
- [6] Jain A. Partial differential equations and finite-difference methods in image Processing, Part 1: Image representation. Journal of optimization Theory and Applications, 1977, 23(1):65-91.
- [7] Koenderink J. The structure of images. Biological Cybernetics, 1984, 50(5):363-370.
- [8] Witkin A. Scale space filtering. Proceedings of the International Joint Conference on Artificial Intelligence, 1983. 1019-1021.
- [9] Alvarez L, Guichard F, Lions P, et al. Axioms and fundamental equations of image processing, Archive for Rational Mechanics and Analysis, 1993, 123(3): 199 - 257.
- [10] Perona P, Malik J. Scale space and edge detection using anisotropic diffusion. IEEE Transactions on pattern Analysis and Machine Intelligence, 1990, 12(7):629--39.
- [11] Guichard F, Morel J. Image Analysis and PDE's. 2001
- [12] Tikhonov A N, Arsenin V Y. Solutions of ill-posed problems. Washington, D. C.: Winston & Sons, 1977.
- [13] Rudin L, Osher S. Total variation based image restoration with free local constraints. IEEE International Conference In Proceedings of 1st International Conference on Image Processing, 1994 (1): 31-35.

- [14] Chavent G, Kunisch K. Regularization of linear least squares problems by total bounded variation. *ESAIM: Control, Optimisation and Calculus of Variations*, 1997, 2: 359–376.
- [15] Goldstein T, Osher S. The Split Bregman algorithm for  $L^1$  regularized problems. *SIAM Journal on Imaging Sciences*, 2009, 2: 323–343.