2D-DOA Estimation for Co-prime L-shaped Arrays with Propagator Method

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Keywords: Co-prime L-shaped array, DOA estimation, propagator method (PM), phase ambiguity.

Abstract. In this paper we propose a DOA estimation method by combining propagator method (PM) algorithm for co-prime L-shaped arrays. The array contains two uniform L-shaped arrays with the minimum inter-element spacing larger than the half-wavelength, which will cause phase ambiguity. Utilizing the co-prime relationship between each decomposed array can eliminate phase ambiguity. With PM algorithm, we avoid peak searching and eigenvalue decomposition of received signal covariance matrix, which have a low computational complexity. Compared with the typical L-shaped arrays which have the same amount of array elements, the proposed method for co-prime L-shaped arrays have a better 2D-DOA estimation performance and a low computational cost. Extensive simulation results demonstrate the effectiveness of the proposed method.

I. Introduction

Direction of Arrival (DOA) estimation, which is a basic problem of array signal processing, is widely used in civil and military realms [1-2]. Numerous 2D-DOA estimation algorithms have been proposed in the past decades for different arrays [3-5]. Among this, the L-shaped array which has a simple structure and easy to be implemented, is widely used in the fields of 2D-DOA estimation. Recently, in order to break the limit of half-wavelength, the notation of co-prime arrays is presented which makes it possible to increase the degree of freedom and enhance the resolution of the array. IN [6], authors present the design approach of co-prime array to eliminate angle ambiguity. In [7], a co-prime linear sparse array, which is capable of significantly increasing the degrees of freedom, is employed for one-dimensional DOA estimation. In [8], authors propose a sparse co-prime L-shaped array, and utilize the array extending capability of FOCs and the co-prime relationship between the interelement spacing to realize DOA estimation without phase ambiguity.

In this paper, we propose a 2D-DOA estimation method for co-prime L-shaped arrays with PM algorithm. We first construct a co-prime L-shaped array consisting of two uniform L-shaped sub-arrays. Then realize DOA estimation through decomposed sub-arrays, respectively. At last, exploiting the co-prime relationship between each decomposed array can eliminate phase ambiguity. With PM algorithm, we avoid peak searching and eigenvalue decomposition of covariance matrix, which have a low computational complexity and obtain great 2D-DOA estimation performance.

The reminder of this paper is organized as follows. Section II introduces the co-prime L-shaped array system model. In Section III, we make the derivation for the 2D-DOA estimation with PM algorithm and give the method to eliminate phase ambiguity. Section IV gives complexity analysis and advantages. Simulation results are provided in Section V and Section VI concludes this paper.

Notations: Lower-case (upper-case) bold symbols to denote vectors (matrices). $(\bullet)^{\boldsymbol{T}}, (\bullet)^{\boldsymbol{H}}, (\bullet)^{-1}, (\bullet)^{+}$ denote the transpose, the conjugate transpose, the inverse and the pseudo inverse, respectively. $\boldsymbol{E}(\bullet)$ is the statistical expectation operator. $angle(\bullet)$ denotes phase operator and $\lceil \bullet \rceil$ denotes ceiling function.

II. Data model

The co-prime L-shaped arrays are nest arrays which can be decomposed into two uniform L-shaped sub-arrays with $2M_i - 1(i=1,2)$ sensor elements, respectively, where M_i is the number of elements in x-direction and y-direction of the *i*th sub-array. M_1 and M_2 are co-prime integers. The array with $2M_1 - 1$ and $2M_2 - 1$ elements has inter-element spacing of $d_1 = M_2\lambda/2$, $d_2 = M_1\lambda/2$, where λ denotes the wavelength. The locations of the elements in the *i*th L-shaped sub-array are in the set:

$$Ls_1 = \{(0, md_1), 0 \le m \le M_1 - 1\} \cup \{(md_1, 0), 0 \le m \le M_1 - 1\} \tag{1}$$

$$Ls_2 = \{(0, nd_2), 0 \le n \le M_2 - 1\} \cup \{(nd_2, 0), 0 \le n \le M_2 - 1\} \tag{2}$$

Fig.1 shows the co-prime L-shaped arrays with $M_1 = 5, M_2 = 7$. θ_k is the elevation angle and φ_k is the azimuth angle of the *k*th source $(\theta_k \in [0, \pi/2], \phi_k \in [0, \pi])$.

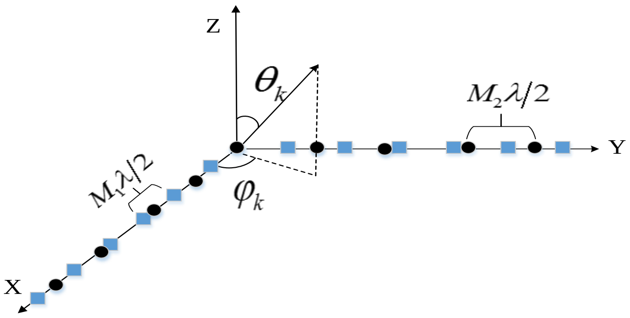

Fig.1. Co-prime L-shaped array model when $M_1 = 5, M_2 = 7$

We consider the uniform L-shaped sub-arrays with $2M - 1$ sensor elements. The observed signals at the sub-array along the x-axis and y-axis are given by

$$\mathbf{X} = \mathbf{A}_x\mathbf{S} + \mathbf{N}_x \quad \text{and} \quad \mathbf{Y} = \mathbf{A}_y\mathbf{S} + \mathbf{N}_y \tag{3}$$

where $\mathbf{S} = [\mathbf{s}_1(t), \mathbf{s}_2(t), \cdots \mathbf{s}_K(t)]^T$, represent the source matrix. $\mathbf{N}_x \in \mathbb{C}^{M \times L}$, $\mathbf{N}_y \in \mathbb{C}^{M \times L}$ represent the additive white Gaussian noise matrix. $\mathbf{A}_x = [\mathbf{a}_x(\theta_1, \varphi_1), \mathbf{a}_x(\theta_2, \varphi_2), \cdots \mathbf{a}_x(\theta_K, \varphi_K)]$ represent steering matrix at x-axis, $\mathbf{A}_y = [\mathbf{a}_y(\theta_1, \varphi_1), \mathbf{a}_y(\theta_2, \varphi_2), \cdots \mathbf{a}_y(\theta_K, \varphi_K)]$ represent steering matrix at y-axis, where $\mathbf{a}_x(\theta_k, \varphi_k) = [1, e^{(-j2\pi d\mu_k/\lambda)}, \cdots e^{-j2\pi d(M-1)\mu_k/\lambda}]$ and $\mathbf{a}_y(\theta_k, \varphi_k) = [1, e^{(-j2\pi d\nu_k/\lambda)}, \cdots e^{-j2\pi d(M-1)\nu_k/\lambda}]$ are the *x*-direction and *y*-direction steering vectors. $\mu_k = \sin\theta_k \cos\varphi_k, \nu_k = \sin\theta_k \sin\varphi_k \ (k = 1, 2, \cdots K)$.

III. 2D PM algorithm and Phase ambiguity elimination

In this section, we first apply PM algorithm to sub-array to obtain the DOA estimations.Then verify that the phase ambiguity can be eliminated by combining the estimation results of the two sub-arrays.

A. PM algorithm

According to [5],we can construct the matrix $\mathbf{C}$ as shown below

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \\ \mathbf{C}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{x1}\mathbf{R}_S\mathbf{A}_{y_1}^H \\ \mathbf{A}_{x1}\mathbf{\Phi}_x\mathbf{R}_S\mathbf{A}_{y_1}^H \\ \mathbf{A}_{x1}\mathbf{\Phi}_y^H\mathbf{R}_S\mathbf{A}_{y_1}^H \\ \mathbf{A}_{x1}\mathbf{\Phi}_x\mathbf{\Phi}_y^H\mathbf{R}_S\mathbf{A}_{y_1}^H \end{bmatrix} = \mathbf{A}\mathbf{R}_S\mathbf{A}_{y1}^H \tag{4}$$

where $\mathbf{A}_{x1}$, $\mathbf{A}_{y1}$ are the first $M-1$ lows of the matrix $\mathbf{A}_x$ and $\mathbf{A}_y$, respectively. $\mathbf{R}_S = E\{\mathbf{S}\mathbf{S}^H\}$, $\mathbf{\Phi}_x = diag[e^{-j2\pi d\mu_1/\lambda}, e^{-j2\pi d\mu_2/\lambda}, \cdots e^{-j2\pi d\mu_k/\lambda}]$, $\mathbf{\Phi}_y = diag[e^{-j2\pi d\nu_1/\lambda}, e^{-j2\pi d\nu_2/\lambda}, \cdots e^{-j2\pi d\nu_k/\lambda}]$.

Partition the matrix $\mathbf{A}$

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_2\end{bmatrix}=\begin{bmatrix}\mathbf{A}_1\\ \mathbf{P}^H\mathbf{A}_1\end{bmatrix}=\begin{bmatrix}\mathbf{I}\\ \mathbf{P}^H\end{bmatrix}\mathbf{A}_1 \tag{5}$$

where $\mathbf{A}_1\in\mathbb{C}^{K\times K}$, $\mathbf{A}_2\in\mathbb{C}^{(4M-4-K)\times K}$. $\mathbf{P}$ is propagator operator. Partition $\hat{\mathbf{R}}_C=\mathbf{C}\mathbf{C}^H$ as

$$\hat{\mathbf{R}}_C=[\hat{\mathbf{G}},\hat{\mathbf{H}}] \tag{6}$$

where $\hat{\mathbf{G}}\in\mathbb{C}^{(4M-4)\times K},\hat{\mathbf{H}}\in\mathbb{C}^{(4M-4)\times(4M-4-K)}$. Propagation operator $\mathbf{P}$ can be estimated by

$$\hat{\mathbf{P}}=(\hat{\mathbf{G}}^H\hat{\mathbf{G}})^{-1}\hat{\mathbf{G}}^H\hat{\mathbf{H}} \tag{7}$$

For Eqs. (4), (5), in the no-noise case, define matrix $\mathbf{E}$ as

$$\mathbf{E}=\begin{bmatrix}\mathbf{I}\\ \hat{\mathbf{P}}^{\mathbf{H}}\end{bmatrix}=\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\\ \mathbf{E}_4\end{bmatrix}=\begin{bmatrix}\mathbf{A}_{x1}\\ \mathbf{A}_{x1}\mathbf{\Phi}_x\\ \mathbf{A}_{x1}\mathbf{\Phi}_y^H\\ \mathbf{A}_{x1}\mathbf{\Phi}_x\mathbf{\Phi}_y^H\end{bmatrix}\mathbf{T} \tag{8}$$

where $\mathbf{T}=\mathbf{A}_1^{-1}$. For Eq. (8), we get

$$\begin{bmatrix}\mathbf{E}_3\\ \mathbf{E}_4\end{bmatrix}=\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\end{bmatrix}\mathbf{T}^{-1}\mathbf{\Phi}_y^H\mathbf{T}\ \text{ and }\ \begin{bmatrix}\mathbf{E}_2\\ \mathbf{E}_4\end{bmatrix}=\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_3\end{bmatrix}\mathbf{T}^{-1}\mathbf{\Phi}_x\mathbf{T} \tag{9}$$

Define $\mathbf{\Omega}_y=\mathbf{T}^{-1}\mathbf{\Phi}_y^H\mathbf{T},\mathbf{\Omega}_x=\mathbf{T}^{-1}\mathbf{\Phi}_x\mathbf{T}$. The least squares solution of $\mathbf{\Omega}_y$ and $\mathbf{\Omega}_x$ are

$$\hat{\mathbf{\Omega}}_y=\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\end{bmatrix}^+\begin{bmatrix}\mathbf{E}_3\\ \mathbf{E}_4\end{bmatrix}\text{ and }\ \hat{\mathbf{\Omega}}_x=\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_3\end{bmatrix}^+\begin{bmatrix}\mathbf{E}_2\\ \mathbf{E}_4\end{bmatrix} \tag{10}$$

We use eigenvalue decomposition (EVD) for $\hat{\mathbf{\Omega}}_y,\hat{\mathbf{\Omega}}_x$ to get

$$\hat{v}_k=\text{angle}(\lambda_k)\lambda/2\pi d\ \text{ and }\ \hat{\mu}_k=-\text{angle}(\varepsilon_k)\lambda/2\pi d \tag{11}$$

where λ_k,ε_k is the kth eigenvalue of $\mathbf{\Omega}_y$ and $\mathbf{\Omega}_x$. After paired $(\hat{\mu}_k,\hat{v}_k)$, $(\hat{\theta}_k,\hat{\varphi}_k)$ can be obtained as

$$\hat{\theta}_k=\sin^{-1}(\sqrt{\hat{\mu}_k^2+\hat{v}_k^2})\ \text{ and }\ \hat{\varphi}_k=\tan^{-1}(\hat{v}_k/\hat{\mu}_k) \tag{12}$$

B. Phase ambiguous elimination

Consider that one single source imping on the co-prime array located at (θ_k,φ_k).The phase difference of adjacent receive signal along x-axis and y-axis can be expressed as

$$\Delta\psi_x=2\pi d\sin\theta_k\cos\varphi_k/\lambda-2k_x\pi\ \text{ and }\ \Delta\psi_y=2\pi d\sin\theta_k\sin\varphi_k/\lambda-2k_y\pi \tag{13}$$

where k_x,k_y are integers, $\Delta\psi_x\in(-\pi,\pi),\Delta\psi_y\in(-\pi,\pi)$. Because $\theta_k\in[0,\pi/2],\varphi_k\in[0,\pi]$, we get $-1<\sin\theta_k\cos\varphi_k<1, 0<\sin\theta_k\sin\varphi_k<1$. $0<(\sin\theta_k\cos\varphi_k)^2+(\sin\theta_k\sin\varphi_k)^2<1$. So we can get

$$k_x\in\left(-\frac{d}{\lambda}-\frac{\Delta\psi_x}{2\pi},\frac{d}{\lambda}-\frac{\Delta\psi_x}{2\pi}\right)\ \text{ and }\ k_y\in\left(-\frac{\Delta\psi_x}{2\pi},\frac{d}{\lambda}-\frac{\Delta\psi_y}{2\pi}\right) \tag{14}$$

Consider $d=M\lambda/2$, the numbers of possible k_x and k_y values equal to M and $\lceil M/2\rceil$. There exsit many different values of 2D DOAs satisfy Eqs.(13). That is the reason why appear phase ambiguity. Fig.2 and Fig.3 shows perfect DOA and its ambiguous DOAs in the transformation domain with the setting of $(\theta_1,\varphi_1)=(20^\circ,30^\circ)$ and $(\theta_2,\varphi_2)=(40^\circ,50^\circ)$.

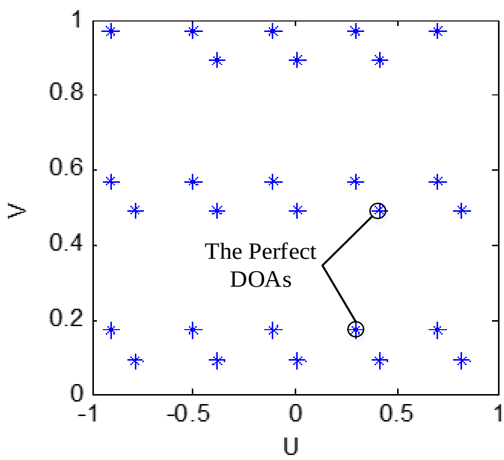

Fig.2. DOA estimations when $d=5\lambda/2$

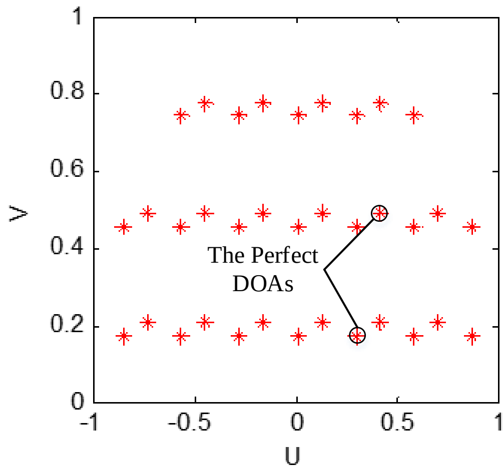

Fig.3. DOA estimations when $d=7\lambda/2$

For the Eqs.(13), the realationship between perfect DOA $(\theta_{p,k},\varphi_{p,k})$ and ambiguous DOAs $(\theta_{a,i,k},\varphi_{a,i,k})$ of ith sub-array are given by

$$\mu_{p,k}-\mu_{a,i,k}=2k_{i,x}/M_j \quad \text{and} \quad \nu_{p,k}-\nu_{a,i,k}=2k_{i,y}/M_j \qquad (15)$$

where $k_{i,x},k_{i,y}$ are integers, M_j is the number of sensor elements at x-axis, $i,j=1,2$ and $j\neq i$.

Suppose that there exist two distinct 2D DOAs, $(\hat{\theta}_{a,1},\hat{\varphi}_{a,1})$, $(\hat{\theta}_{a,2},\hat{\varphi}_{a,2})$ that are both obtained by the two sub-arrays which are both ambiguous 2D DOAs with respect to the perfect 2D DOA $(\theta_{p,k},\varphi_{p,k})$. According to Eqs.(15), we can get

$$\hat{\mu}_{a,1}-\hat{\mu}_{a,2}=2k_{i,x}/M_j \quad \text{and} \quad \hat{\nu}_{a,1}-\hat{\nu}_{a,2}=2k_{i,y}/M_j \qquad (16)$$

where $k_{i,x}$ and $k_{i,y}$ are integers and $k_{i,x}$ is in the range of $(-M_j,M_j)$, while $k_{i,y}$ is in the range of $(-M_j/2,M_j/2)$ and $i,j=1,2$ and $j\neq i$.Hence,we get

$$\frac{k_{1,x}}{M_2}=\frac{k_{2,x}}{M_1} \quad \text{and} \quad \frac{k_{1,y}}{M_2}=\frac{k_{2,y}}{M_1} \qquad (17)$$

Due to the co-prime property between M_1 and M_2, only $k_{1,x}=k_{2,x}=0$, $k_{1,y}=k_{2,y}=0$ that satisfy the Eq.(17). Hence, combining two DOA estimations between sub-arrays, there exists and uniquely exists a common 2D DOA, which is the perfect DOA.

IV. Performance analysis

A. Complexity

The complexity of the proposed method is $O\left(6K^3+4K^2(M_1+M_2-2)+16K\left((M_1-1)^2+(M_2-1)^2\right)+16\left((M_1-1)^3+(M_2-1)^3\right)\right)$

B. Advantage

a. The method avoid spectrum peak search and don't require eigenvalue decomposition of received signal covariance matrix. It has lower computational complexity than standard L-shaped arrays.

b. The co-prime L-shaped arrays have better performance than the uniform L-shaped arrays with the same number of sensor elements when using PM algorithm.

V. Simulation results

The co-prime L-shaped array can be decomposed into two uniform sub-array with $2M_1-1$ and $2M_2-1$ sensor elements. Here, we set $M_1=5, M_2=7$.Suppose K=2 sources impinging on the arrays located at $(20°,30°)$ and $(40°,50°)$.

The root mean square error (RMSE) of the estimations is defined as the performance metric:

$$RMSE = \sqrt{\frac{1}{SK}\sum_{s=1}^{S}\sum_{k=1}^{K}\left(\alpha_k - \hat{\alpha}_{k,s}\right)^2} \qquad (18)$$

where S denotes the times of Monte-Carlo simulations and $\hat{\alpha}_{k,s}$ is the estimation of the kth angle α_k for the sth trial (S=1000).

In this simulation, we first give the simulation results when $SNR = 5dB$ and $15dB$, where $L = 200$ as shown in Fig.4 and Fig.5. Second, we study the RMSE performance of the co-prime L-shaped array and the uniform L-shaped array under different SNRs as shown in Fig.6. It is clearly indicated that the co-prime L-shaped array has better performance. Last, we study the RMSE performance of the co-prime L-shaped array in the setting with different number of snapshots L, as shown in Fig.7. It is clearly indicated that the performance of the co-prime L-shaped array is getting better with L increasing.

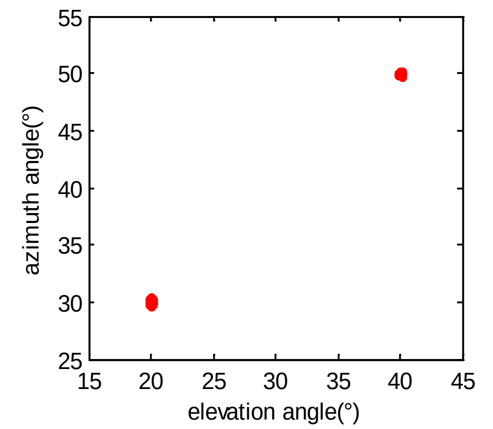

Fig.4. Scatter diagram at SNR=5dB

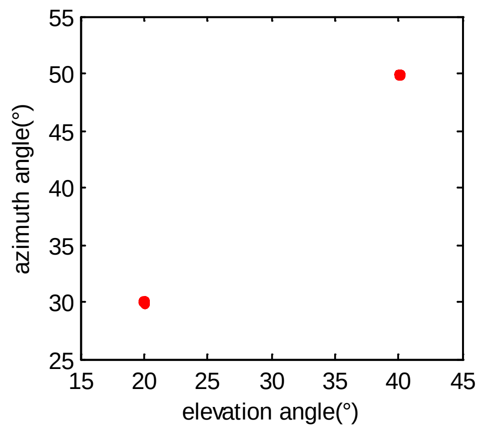

Fig.5. Scatter diagram at SNR= 15dB.

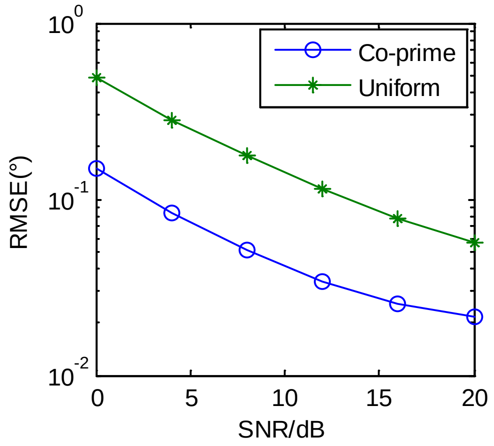

Fig.6. RMSE versus the SNR

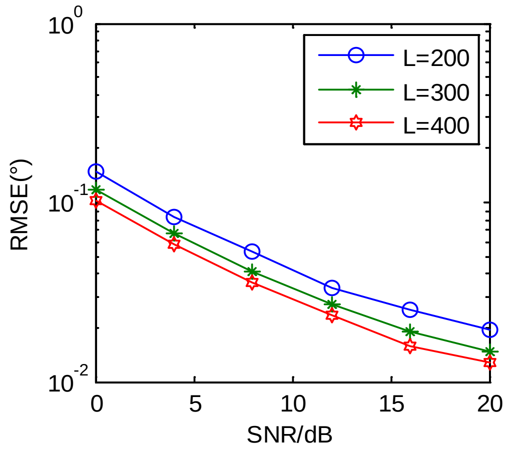

Fig.7. RMSE versus the SNR in *L*

VI. Summary

In this paper, we have investigated the problem of 2D-DOA estimations in co-prime L-shaped arrays whose inter-element spacing is larger than half wavelength, which will cause phase ambiguity. To avoid spectrum peak search, we estimate the 2D DOA with PM algorithm. By combining the results of the two sub-arrays, the phase ambiguity can be eliminated, so we can obtain the perfect DOAs. It is shown that the co-prime L-shaped array has better performance and lower complexity than the typical uniform L-shaped array through simulations.

References

[1] Zhang X F, Wang F, Xu D Z, "Theory and application of array signal processing," Beijing National Defence Industry Press, 2010.

[2] Liu D S. "Spatial spectrum estimation and its applications," Anhui: University of Science and Technology of China Press, 1997.

[3] Schmidt R, "Multiple emitter location and signal parameter estimation". IEEE Transactions on Antennas and Propagation, vol.34, no.3, pp: 276-280. 1986.

[4] Roy R, Kailath T, "ESPRIT-estimation of signal parameters via rotational invariance techniques," IEEE Transactions on Acoustics, Speech and Signal Processing, vol.37, no.7, pp: 984-995, 1989.

[5] Zhang X, Gao X, Chen W, "Improved blind 2D-direction of arrival estimation with L-shaped array using shift invariance property," Journal of Electromagnetic Waves and Applications, vol.23, no.5-6, pp: 593-606, 2009.

[6] Vaidyanathan P P, Pal P, "Sparse sensing with co-prime samplers and arrays," IEEE Transactions on Signal Processing, vol.59, no.2, pp: 573-586, 2011.

[7] Pal P, Vaidyanathan P P, "Nested arrays: a novel approach to array processing with enhanced degrees of freedom," IEEE Transactions on Signal Processing, vol.58, no.8, pp: 4167-4181, 2010.

[8] Shao H, Su W M, et al, "Two-dimensional direction finding using a sparse co-prime L-shaped array," Chinese Journal of Radio Science, vol.27, no.5, pp: 886-891, 2012.