Existence of Solution to Generalized Multivalued Vector Variational-like Inequalities

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***Abstract*—In this paper, we consider a generalized multivalued vector variational-like inequality and obtain some existence results. some special cases are also discussed.**

Keywords—Generalized multivalued vector variational-like inequality; Existence result; Affine mapping; Lower semicontinuity

I. *Introduction and Preliminaries*

Let Y be a Banach space with a convex cone P such that $\operatorname{int} P \neq \phi$ and $P \neq Y$, where int denotes the interior. We use the following vector ordering: for any $x, y \in Y$, $y < x$ if and only if $y - x \in -\operatorname{int} P$; $y \not< x$ if and only if $y - x \notin -\operatorname{int} P$. Let X be a nonempty subset of a Banach space E and Y a Banach space with a convex cone P such that int $P \neq \phi$ and $P \neq Y$. Let $N : L(E,Y) \times L(E,Y) \times L(E,Y) \to L(E,Y)$ be a single-valued mapping, $M, S, T : X \to 2^{L(E,Y)}$ be three set-valued mappings, where $L(E,Y)$ is the space of all linear continuous mappings from E into Y, and $\langle l, x \rangle$ denotes the value of l at x, $\eta : X \times X \to Y$, $H : X \times X \to Y$ be two bi-mappings. We consider the following two generalized multivalued vector variational-like inequality, for short, denoted by GMVVLI-1 and GMVVLI-2, respectively.

GMVVLI-1: Find $y \in X$, such that for any

$u \in M(y), v \in S(y), w \in T(y)$ satisfying $\langle N(u,v,w), \eta(x,y) \rangle + H(x,y) \not< 0, \ \forall x \in X.$

GMVVLI-2: Find $y \in X$,

such that $\exists u \in M(y), v \in S(y), w \in T(y)$ satisfying $\langle N(u,v,w), \eta(x,y) \rangle + H(x,y) \not< 0, \ \forall x \in X.$

It is easy to see that each solution of GMVVLI-1 is that of GMVVLI-2, but the converse maybe not true in general. The aim of this paper is to derive the existence results of GMVVLI-1. It is well known that GMVVLI-1 and GMVVLI-2 are extensions of the vector variational inequality, which is a generalized form of a variational inequality, having applications in different areas of optimization, optimal control, operations research, economics equilibrium and free boundary value problems, see, for instance, [1-3] and the references therein. Now we give some definitions and lemma needed for the proof of the existence results.

Definition 1.1 [4] Let X be a subset of a topological space E. Then a set-valued mapping $F : X \to 2^E$ is called the KKM mapping if for each finite subset $\{x_1, x_2, \cdots, x_n\}$ of X, co $\{x_1, x_2, \cdots, x_n\} \subset \bigcup_{i=1}^n F(x_i)$, where co$\{x_1, x_2, \cdots, x_n\}$ is the convex hull of $\{x_1, x_2, \cdots, x_n\}$.

Definition 1.2 Let $H : X \times X \to Y$ be a mapping. H is said to be affine in the second argument if, for each $x \in X$, $t_i \in R(i = 1,2,\cdots,n)$ with $\sum_{i=1}^n t_i = 1$ such that for each $y_i \in X_i (i = 1,2,\cdots,n)$, we have $H(x, \sum_{i=1}^n t_i y_i) = \sum_{i=1}^n t_i H(x, y_i)$.

Similarly, we can define the affinity of H in the first argument.

II. Existence results

Theorem 2.1 Let X be a compact convex subset of a Banach space E and Y a Banach space with convex cone P such that int $P \neq \phi$ and $P \neq Y$. Let $N : L(E,Y) \times L(E,Y) \times L(E,Y) \to L(E,Y)$ be a continuous mapping. Suppose that:

(i) $M, S, T : X \to 2^{L(E,Y)}$ are lower semicontinuous;

(ii) $H : X \times X \to Y$ and $\eta : X \times X \to Y$ are continuous in the second argument, respectively, and $H(x,x) = \eta(x,x) = 0$ for all $x \in X$;

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(iii) the multivalued mapping $w: X \to 2^Y$ defined by $W(x) = Y / \{-\operatorname{int} P\}$, has a closed graph in $X \times Y$;

(iv) for each $y \in X$,

$$B_y = \{x \in X : \exists u \in M(y), v \in S(y), w \in T(y)$$

$$such \quad that \ \langle N(u,v,w), \eta(x,y)\rangle + H(x,y) < 0 \ \}$$

is convex. Then GMVVLI-1 is solvable.

Proof. Define a multivalued mapping $F : X \to 2^X$ by

$$F(x) = \left\{ \begin{matrix} y \in X : \langle N(u,v,w), \eta(x,y)\rangle + H(x,y) \not< 0, \\ \forall u \in M(y), u \in S(y), w \in T(y) \end{matrix} \right\}$$

, $\forall x \in X$. We first prove that F is a KKM mapping. Suppose to the contrary, F is not a KKM mapping. Then the convex hull of every finite subset $\{x_1, x_2, \cdots, x_n\}$ of X is not contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$. Let $y \in co\{x_1, x_{2,}, \cdots, x_n\}$. Then $y = \sum_{i=1}^n \alpha_i x_i$ for some $\alpha_i \geq 0, i = 1,2,\cdots,n$ with $\sum_{i=1}^n \alpha_i = 1$ and $y \notin \bigcup_{i=1}^n F(x_i)$. Then we have for each $i \in \{1,2,\cdots,n\}, \exists u \in M(y), v \in S(y), w \in T(y)$, such that

$$\langle N(u,v,w), \eta(x_i, y)\rangle + H(x_i, y) < 0 .$$

Since by assumption (iv), B_y is convex, then $\operatorname{co}\{x_1, x_2, \cdots, x_n\} \subset B_y$,

$$\left\langle N(u,v,w), \eta(\sum_{i=1}^n \alpha_i x_i)\right\rangle + H(\sum_{i=1}^n \alpha_i x_i, \sum_{i=1}^n \alpha_i x_i) \in -\operatorname{int} P$$

It follows from assumption (ii) that $0 \in -\operatorname{int} P$, which contradicts $P \neq Y$. Therefore F is a KKM mapping. Next we prove that for any $x \in X$, $F(x)$ is closed. Indeed, Let $\{y_n\}$ be s sequence in $F(x)$ converging to $y^* \in X$. By the lower semicontinuity of M, S, T, for $\forall (u^*, v^*, w^*) \in M(y^*) \times S(y^*) \times T(y^*)$, the re exist $u_n \in M(y_n), v_n \in S(y_n), w_n \in T(y_n)$ for all n such that

$$(u_n, v_n, w_n) \to (u^*, v^*, w^*) \in L(E,Y) \times L(E,Y) \times L(E,Y)$$

Since $y_n \in F(x)$ for all n, we have

$$\langle N(u_n, v_n, w_n), \eta(x_n, y_n)\rangle + H(x, y_n) \not< 0,$$

Which implies that

$$\langle N(u_n, v_n, w_n), \eta(x_n, y_n)\rangle + H(x, y_n) \in W(y_n).$$

It follows from assumption (ii) that H and η are continuous in the second argument, respectively, and note that N is continuous, we has a closed graph in $X \times Y$ and $(u_n, v_n, w_n, y_n) \to (u^*, v^*, w^*, y^*)$, we get

$$\langle N(u_n, v_n, w_n), \eta(x, y_n) + H(x, y_n) + H(x, y_n)\rangle$$

$$\to \langle N(u^*, v^*, w^*), \eta(x, y^*)\rangle + H(x, y^*) \in W(y^*) .$$

Meaning that $\langle N(u^*, v^*, w^*), \eta(x, y^*)\rangle + H(x, y^*) \not< 0$. Thus $y^* \in F(x)$ and $F(x)$ is closed. Furthermore, since X is a compact subset of E and $F(x) \subset X$ for each $x \in X$. Then $F(x)$ is compact. It follows from Lemma 1.1 that $\bigcap_{x \in X} F(x) \neq \phi$, that is, there exists $y \in X$ such that for any $(u,v,w) \in M(y) \times S(y) \times T(y)$, satisfying

$$\langle N(u,v,w), \eta(x,y)\rangle + H(x,y) \not< 0, \forall x \in X ,$$

which implies that GMVVLI-1 is solvable. This completes the proof.

Corollary 2.1 Let X be a compact convex subset of a Banach space E and Y a Banach space with convex cone P such that, int $P \neq \phi$ and $P \neq Y$.

$N : L(E,Y) \times L(E,Y) \times L(E,Y) \to L(E,Y)$ be a continuous mapping. Assume that

(i) $M, S, T : X \to 2^{L(E,Y)}$ are lower semicontinuous;

(ii) $H : X \times X \to Y, \eta : X \times X \to Y$ are continuous in the second argument and affine in the first argument, respectively, and $H(x,x) = \eta(x,x) = 0$ for all $x \in X$;

(iii) the multivalued mapping $W : X \to 2^Y$ defined by $W(x) = Y / \{-\operatorname{int} P\}$, has a closed graph in $X \times Y$.

Then GMVVLI-1 is solvable.

Proof It is sufficient to prove that for each $y \in X$, the set

$$B_y = \{x \in X : \exists u \in M(y), v \in S(y), w \in T(y)$$

$$such \quad that \ \langle N(u,v,w), \eta(x,y)\rangle + H(x,y) < 0 \ \}$$

is convex. For this, let $x_1, x_2 \in By$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. Then for some $u \in M(y)$, $v \in S(y)$, $w \in T(y)$, we have

$$\langle N(u,v,w), \eta(x_1, y)\rangle + H(x_1, y) \in -\operatorname{int} P, \quad (1)$$

and

$$\langle N(u,v,w),\eta(x_2,y)\rangle + H(x_2,y) \in -\operatorname{int} P. \qquad (2)$$

Nothing that –int P is a convex cone and multiplying (1) by α and (2) by β and adding, we have

$$\alpha\langle N(u,v,w),\eta(x_1,y)\rangle + H(x_1,y) + \beta\langle N(u,v,w),\eta(x_2,y)\rangle + H(x_2,y)$$
$$\in -(\alpha \operatorname{int} P + \beta \operatorname{int} P) = -\operatorname{int} P. \qquad (3)$$

It follows from the affinity of H and η in the first argument and (3) that

$$\langle N(u,v,w),\alpha\eta(x_1,y)\rangle + \langle N(u,v,w),\beta\eta(x_2,y)\rangle$$
$$+ \alpha H(x,y) + \beta H(x_2,y)$$
$$= \langle N(u,v,w),\eta(\alpha x_1 + \beta x_2,y)\rangle + H(\alpha x_1 + \beta x_2,y) \in -\operatorname{int} P.$$

That is, $\alpha x_1 + \beta x_2 \in By$, thus B_y is convex, completing the proof.

Remark 2.1 Theorem 2.1 and corollary 2.1 extend the corresponding results of [5].

Remark 2.2 If all the conditions of Theorem 2.1 and corollary 2.1 are satisfied, respectively, then GMVVLI-2 is solvable.

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