Numerical analysis of the elliptic optimal control problem with L^p-norm state constraints

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***Abstract*—An optimal control problem governed by an elliptic equations with an L^p-norm state constraint is analyzed. The finite element approximation and its optimal a priori estimates are given. The numerical experiments are performed to confirm the a priori estimates.**

Keywords: elliptic control, state constraint, finite element method, error estimates.

I. INTRODUCTION

Optimal control problems are crucial to many engineering applications, there have been extensive studies for the finite element approximation of these problems in literatures, most of which focused upon control-constrained problem. The state constrained control problem is also frequently met in practical applications of the real-life. It has aroused many people's interests, Casas in [3] discusse the pointwise state constrained one, for the finite element approximation the readers may refer to [8] and references cited therein. Later, some people discuss the regularized problem, namely, the state is considered in the L^2 topology with the pointwise constraint almost everywhere, it is well known that the Lagrange multipliers are regular and one may avoid some difficulties occurred in the pointwise constrained problem. As well known, the integral or the energy of the state are worth concerning in many control problems. For example, one probably want to constrain the concentration, the temperature in the average sense, or the kinetic energy of the flow, etc. And the optimal control problem with a state energy constraint is often used to study the noise removal in image processing, see, for instance, [12] and references cited therein. In [4], a more general problem is studied and the corresponding finite element approximation and error estimates are given. Recently, Liu, Yang and Yuan in [11] derive the optimal order finite element error estimates of the integral state constrained problem and propose a gradient projection algorithm to approximate its solution. Then, Yuan and Yang discussed the control problem with an L^2-norm state constraint in [15]. Meanwhile, Deng and his colleagues study the topology optimization of steady and unsteady Navier-Stokes flows in [9], [10].

This paper proceeds as follows. In section II, we introduce the model problem and construct the finite element approximation. Further, we give the corresponding optimality conditions. In section III, we derive the a priori error estimates. In the end of this article, some numerical tests are presented to verify the a priori estimation results.

II. MODEL PROBLEM AND FINITE ELEMENT APPROXIMATION

Let Ω be a bounded, convex and connetive domain in $\mathbb{R}^d$, $1 \leqslant d \leqslant 3$, with the Lipschitz boundary $\partial\Omega$. Throughout this paper we use the standard notation for the Sobolev spaces, norms and seminorms. For example, we denote $H^m(\Omega)$ the function space $W^{m,2}(\Omega)$ as the usual notation[1] where m is an integer and $H_0^1(\Omega)$ the subspace of $H^1(\Omega)$ with the homogeneous trace. For the real numbers $1 \leq p', q' \leq +\infty$, $\frac{1}{p'} + \frac{1}{q'} = 1$, we denote the dual inner product by

$$(v, w) = \int_\Omega vw, \quad \forall v \in L^{p'}(\Omega),\ w \in L^{q'}(\Omega),$$

and the norm in $W^{k,p'}(\Omega)$ by $\|\cdot\|_{k,p';\Omega} = \|\cdot\|_{W^{k,p'}(\Omega)}$.

Let us consider the differential operator

$$Aw = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}(a_{ij}\frac{\partial w}{\partial x_i}) + bw, \tag{1}$$

where

$$\begin{aligned} & a_{ij} = a_{ji} \text{ is Lipschitz on } \bar{\Omega} && \forall 1 \le i, j \le d, \\ & \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \ge a_0|\xi|^2, \quad a_0 > 0 && \forall \xi \in \mathbb{R}^d, \\ & b \in L^\infty(\Omega), \quad b(x) \ge 0 && a.e.\ x \in \Omega. \end{aligned} \tag{2}$$

In this paper we investigate the following optimal control problem

$$(\mathscr{P}) \qquad \begin{aligned} & \min_{v} \quad \mathcal{J}(v, z) = \frac{1}{2}\int_\Omega (z - y_d)^2 + \frac{\alpha}{2}\int_\Omega v^2 \\ & s.t. \\ & \begin{cases} Az = f + v & in\ \Omega, \\ z = 0 & on\ \partial\Omega, \\ z \in K, \end{cases} \end{aligned}$$

where $f \in L^s(\Omega), s > d$ and the constraint set K is defined as

$$K = \Big\{ w, \|w\|_{0,n_e;\Omega} \leq \gamma \Big\}, \quad n_e \text{ is a nonzero even }, \ \gamma > 0. \tag{3}$$

In addition, we denote c or C the general positive constant that independent of grid parameters h.

A. Optimality conditions

Let us introduce function spaces $U = L^2(\Omega)$ for the control and $Y = H_0^1(\Omega)$ for the state variable. Obviously $K \cap Y$ is a closed convex subspace of Y. As well known, the optimality conditions of problem $\mathscr{P}$ can be derived by a Lagrange functional $\mathscr{L} : U \times \mathbb{R} \to \mathbb{R}$ defined as

$$\mathscr{L}(v, \lambda) = J\big(v, z(v)\big) + \lambda\big(\|z\|^{n_e}_{0,n_e;\Omega} - \gamma^{n_e}\big)/n_e. \tag{4}$$

Then we state the optimality conditions as follows, for the proof the reader may refer to Clarke[6],[7], Casas[4] or Yuan and Yang [15].

Lemma 1: The control u is the solution of problem $\mathscr{P}$ and y is the corresponding optimal state, if and only if there exist $\lambda \in \mathbb{R}$ and $p \in Y$ satisfying the following weak form equations

$$\begin{cases} a(y, v) = (f + u, v) & \forall v \in Y, \\ a(v, p) = (y - y_d + \lambda y^{n_e - 1}, v) & \forall v \in Y, \\ p + \alpha u = 0 & in\ L^2(\Omega), \\ \lambda\Big(\|y\|^{n_e}_{0,n_e;\Omega} - \gamma^{n_e}\Big) = 0,\ \lambda \geq 0. \end{cases} \tag{5}$$

where

$$a(w, v) = \sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_j} + \int_\Omega bwv \quad \forall w, v \in Y.$$

Remark 1: For the regularities of the state and costate, it is obvious that $y \in H^2(\Omega) \cap Y$ since $u \in L^2(\Omega)$, then $p \in H^2(\Omega) \cap Y$ since that $y \in L^\infty(\Omega)$ and $y_d \in L^2(\Omega)$. Furthermore, we know that $y \in W^{2,s}(\Omega)$ for $f \in L^s(\Omega)$.

Letting v take a suitable function in equations (5), we can obtain below expression

$$n_e \lambda \|y\|^{n_e}_{0,n_e;\Omega} = (f + u, p) + (y_d - y, y). \tag{6}$$

B. Finite element discretization

We are ready to study the finite element approximation of the model problem. For simplicity, let us assume Ω be a polygonal domain in the following context. We only consider n-simply element, which are widely used. Let $T^h = \bigcup \tau$ be a quasi-regular triangulation of Ω with maximum mesh size $h := \max_{\tau \in T^h}\{diam(\tau)\}$, in which each element has at most one face on $\partial\Omega$, $\overline{\tau}$ and $\overline{\tau}'$ have either only one common vertex or a whole edge in 2-d case or face in 3-d case if $\overline{\tau}$ and $\overline{\tau}' \in T^h$. Associated with T^h is a finite dimensional subspace of $Y \cap L^\infty(\Omega)$ defined by $Y^h := \{w_h \in C^0(\Omega) : w_h|_\tau$ are polynomials of degree less than and equal to r ($r \geqslant 1$) for each $\tau \in T^h\}$.

In this paper, we only consider the simplest finite element spaces, i.e., $r = 1$, which means that the piecewise linear conforming elements for the control, state and the co-state. Suppose that the finite element space Y_h has the following approximation properties(see [5], for instance)

$$\begin{aligned} \inf_{w_h \in Y^h} \quad & \Big[\|w - w_h\|_{L^2(\Omega)} + h\|\nabla(w - w_h)\|_{L^2(\Omega)}\Big] \\ & \leqslant Ch^2\|w\|_{2,2;\Omega}, \quad \forall\ w \in H_0^1(\Omega) \bigcap H^2(\Omega). \end{aligned} \tag{7}$$

Then the finite element approximation of the optimal control problem ($\mathscr{P}$) reads as

$$(\mathscr{P}_h) \quad \begin{cases} \min\limits_{v \in Y_h} \quad \mathcal{J}(v, z) = \dfrac{1}{2}\displaystyle\int_\Omega (z - y_d)^2 + \dfrac{\alpha}{2}\displaystyle\int_\Omega v^2 \\ \quad s.t. \\ a(z, w) = (f + v, w) \quad \forall w \in Y_h, \quad z \in Y_h \cap K. \end{cases}$$

Similarly, we can obtain the optimality conditions of discretization problem $\mathscr{P}_h$.

Lemma 2: The control u_h is the optimal control of problem $\mathscr{P}_h$ and y_h is the corresponding optimal state, if and only if that there exists $\lambda_h \in \mathbb{R}$ and $p_h \in Y_h$ satisfying

$$\begin{cases} a(y_h, v_h) = (f + u_h, v_h) & \forall v_h \in Y_h, \\ a(v_h, p_h) = (y_h - y_d + \lambda_h y_h^{n_e - 1}, v_h) & \forall v_h \in Y_h, \\ p_h + \alpha u_h = 0, \\ \lambda_h\Big(\|y_h\|^{n_e}_{0,n_e;\Omega} - \gamma^{n_e}\Big) = 0,\ \lambda_h \geq 0. \end{cases} \tag{8}$$

Note that the above discrete optimal state y_h is $W^{1,s}$-regular, and $\|y_h\|_{1,s;\Omega} \leq c(\|f_{0,s;\Omega} + \|u_h\|_{0,s;\Omega})$, see, for instance, [2, Th8.5.3], and it follows from $s > d$ that $\|y_h\|_{0,\infty;\Omega} \leq c(\|f_{0,s;\Omega} + \|u_h\|_{0,s;\Omega})$.

III. A PRIORI ERROR ESTIMATE

In this section, we study the a priori estimates of the finite element approximation. Let us state the main results of this section in the following theorem.

Theorem 1: Let (u, y, p, λ) and $(u_h, y_h, p_h, \lambda_h)$ be the solutions of equations (5) and (8), respectively, then we have

$$\|y - y_h\|_{1,2;\Omega} + \|p - p_h\|_{1,2;\Omega} \leqslant Ch, \tag{9}$$

and the L^2-norm estimate

$$\|u - u_h\|_{0,2;\Omega} + \|y - y_h\|_{0,2;\Omega} + \|p - p_h\|_{0,2;\Omega} + |\lambda - \lambda_h| \leqslant Ch^2. \tag{10}$$

To get those results, we need introduce the auxiliary variables $y_h(u)$ by

$$a(y_h(u), v_h) = (f + u, v_h) \quad \forall v_h \in Y_h, \tag{11}$$

and $p_h(y)$ by

$$a(v_h, p_h(y)) = (y - y_d + \lambda y^{n_e - 1}, v_h) \quad \forall v_h \in Y_h. \tag{12}$$

Before giving the proof of the above theorem, we need to derive some lemmas.

First, it is clear that $y_h(u)$ and $p_h(y)$ are the finite element approximations of y and p, respectively, so we have the standard results, see, for instance, [2] and [6].

$$\begin{aligned} &\|y-y_h(u)\|_{0,2;\Omega}+h\|y-y_h(u)\|_{1,2;\Omega}\le Ch^2,\\ &\|p-p_h(y)\|_{0,2;\Omega}+h\|p-p_h(y)\|_{1,2;\Omega}\le Ch^2; \end{aligned} \tag{13}$$

Further, we have the following estimates.

Lemma 3: Let (u,y) and (u_h,y_h) be the solutions of equations (5) and (8), respectively. Then we have

$$\|y-y_h\|_{0,2;\Omega}+h\|y-y_h\|_{1,2;\Omega}\le C\Big(h^2+\|u-u_h\|_{0,2;\Omega}\Big).$$

Proof: Combining equations (8) and (11), we get

$$a(y_h(u)-y_h,v_h)=(u-u_h,v_h)\quad \forall v_h\in Y_h.$$

Taking $v_h=y_h(u)-y_h$, observing (2), we obtain

$$\|y_h(u)-y_h\|_{1,2;\Omega}\le c\|u-u_h\|_{0,2;\Omega}.$$

Then utilizing the estimates (13), we obtain the lemma. ■

Next, we can bound the error of $|\lambda-\lambda_h|$ and the co-state in the following lemma.

Lemma 4: Let p,λ and p_h,λ_h be the solution of equations (5) and (8), respectively. Then we have

$$|\lambda-\lambda_h|\le C\Big(h^2+\|u-u_h\|_{0,2;\Omega}\Big)$$

and

$$\|p-p_h\|_{0,2;\Omega}+h\|p-p_h\|_{1,2;\Omega}\le C\Big(h^2+\|u-u_h\|_{0,2;\Omega}\Big).$$

Proof: Note for λ we have expression (6), similarly, we also have the expression for λ_h that

$$\lambda_h\|y_h\|^{n_e}_{0,n_e;\Omega}=(f+u_h,p_h)+(y_d-y_h,y_h). \tag{14}$$

Observing the third equation of (5) and of (8), combing the expressions (6) and (14), we can derive as

$$\begin{aligned} &\lambda\|y\|^{n_e}_{0,n_e;\Omega}-\lambda_h\|y_h\|^{n_e}_{0,n_e;\Omega}=(f,p-p_h)+(u_h,p_h)\\ &\quad-(u,p)+(y_d,y-y_h)+(y_h,y_h)-(y,y)\\ &=\Big[(f,u_h-u)+(u+u_h,u-u_h)\Big]+\Big[(y_d,y-y_h)\\ &\quad+(y_h-y,y+y_h)\Big] \end{aligned}$$

Obviously that $\lambda=\lambda_h=0$ is a trivial case, so in what follows we only need to study cases that there is at least one of λ,λ_h is nonzero. Suppose that $\lambda\neq 0$(the case $\lambda_h\neq 0$ is similar), which means that $\|y\|^{n_e}_{0,n_e;\Omega}=\gamma^{n_e}$, from the above equality and Lemma 3 we have

$$\begin{aligned} &|\lambda-\lambda_h|\gamma^{n_e}\\ &=\Big[(f,u_h-u)+(u+u_h,u-u_h)\Big]+\Big[(y_d,y-y_h)\\ &\quad+(y_h-y,y+y_h)\Big]+\lambda_h\Big[\|y_h\|^{n_e}_{0,n_e;\Omega}-\|y\|^{n_e}_{0,n_e;\Omega}\Big]\\ &\le C\Big[\|u_h-u\|_{0,2;\Omega}+\|y-y_h(u)\|_{0,2;\Omega}+h^2\Big]\\ &\le C\Big[\|u_h-u\|_{0,2;\Omega}+h^2\Big], \end{aligned}$$

where we use the fact $y,y_h\in L^\infty(\Omega)$ that

$$\begin{aligned} &\|y_h\|^{n_e}_{0,n_e;\Omega}-\|y\|^{n_e}_{0,n_e;\Omega}=\int_\Omega(y-y_h)\Big(\sum_{i=0}^{n_e-1}C^i_{n_e-1}y^iy_h^{n_e-1-i}\Big)\\ &\le c\Big(\|y\|_{0,\infty;\Omega}+\|y_h\|_{0,\infty;\Omega}\Big)\|y-y_h\|_{0,2;\Omega}. \end{aligned}$$

To prove the second inequality of the lemma, we subtract the second equation of equations (8) from equation (12) to get

$$a(p_h(y)-p_h,v_h)=(y-y_h+\lambda y^{n_e}-\lambda_h y_h^{n_e},v_h)\quad \forall v_h\in Y_h,$$

which combining the strong elliptic property of operator $a(\cdot,\cdot)$ in Y yields

$$\|p_h(y)-p_h\|_{1,2;\Omega}\le c\|y-y_h+\lambda y^{n_e}-\lambda_h y_h^{n_e}\|_{0,2;\Omega}.$$

Similar to the proof of $|\lambda-\lambda_h|$ and using the above results we can obtain

$$\|p_h(y)-p_h\|_{1,2;\Omega}\le c\Big[h^2+\|u-u_h\|_{0,2;\Omega}+\|y-y_h\|_{0,2;\Omega}\Big],$$

Combing the above results and (13) and Lemma 3, we can easily get the second inequality of the lemma, thus the proof is completed. ■

Then we can derive the error estimate of the control variables now.

Lemma 5: Assume that u and u_h are the optimal control of problems $\mathscr{P}$ and $\mathscr{P}_h$, respectively. Then

$$\|u-u_h\|_{0,2;\Omega}\le Ch^2$$

holds for h small enough.

Proof: Combining equation (11), (12) and equations (8), we have($\forall v_h\in Y_h,\ \forall w_h\in Y_h$)

$$\begin{aligned} &a(y_h(u)-y_h,v_h)=(u-u_h,v_h),\\ &a(p_h(y)-p_h,w_h)=(y-y_h+\lambda y^{n_e-1}-\lambda_h y_h^{n_e-1},w_h). \end{aligned}$$

Taking $v_h=p_h(y)-p_h$ and $w_h=y_h(u)-y_h$ in the above, we have

$$\begin{aligned} &\alpha\|u-u_h\|^2_{0,2;\Omega}=(u-u_h,p_h-p_h(y))+(u-u_h,p_h(y)-p)\\ &=-(y-y_h+\lambda y^{n_e-1}-\lambda_h y_h^{n_e-1},y_h(u)-y_h)\\ &\quad+(u-u_h,p_h(y)-p)\\ &=-(y-y_h+\lambda y^{n_e-1}-\lambda_h y_h^{n_e-1},y-y_h)-(y-y_h+\lambda y^{n_e-1}\\ &\quad-\lambda_h y_h^{n_e-1},y_h(u)-y)+(u-u_h,p_h(y)-p)\\ &\le-\|y-y_h\|_{0,2;\Omega}+(\lambda y^{n_e-1}-\lambda_h y_h^{n_e-1},y_h-y)\\ &\quad+ch^2\big(\|y-y_h\|_{0,2;\Omega}+|\lambda-\lambda_h|+\|u-u_h\|_{0,2;\Omega}\big)\\ &\le ch^2\big(h^2+\|u-u_h\|_{0,2;\Omega}\big), \end{aligned}$$

where we use that $y,y_h\in L^\infty(\Omega)$ again. Thus we have $\|u-u_h\|_{0,2;\Omega}\le Ch^2$, so the lemma is proved. ■

Finally, substituting the results of Lemma 5 into Lemma 3-4, we can obtain the proof of Theorem 1.

TABLE I
NUMERICAL RESULTS OF EXPERIMENT

Mesh	mesh 1	mesh 2	mesh 3	mesh 4
h	0.1	0.05	0.025	h=0.0125
#Nodes	139	513	1969	7713
#Elements	236	944	3776	15104
#Edges	374	1456	5744	22816
$\|y-y_h\|_0$	3.9590e-03	9.9611e-04	2.4962e-04	6.2454e-05
$\|p-p_h\|_0$	1.6588e-01	4.1552e-02	1.0407e-02	2.6036e-03
$\lvert\lambda-\lambda_h\rvert$	1.7150e+00	4.2320e-01	1.0559e-01	2.6393e-02
$\|u-u_h\|_0$	1.6588e-01	4.1552e-02	1.0407e-02	2.6036e-03
$\|y-y_h\|_1$	3.2767e-01	1.6392e-01	8.2008e-02	4.1014e-02
$\|p-p_h\|_1$	1.5039e+00	6.9957e-01	3.4309e-01	1.7071e-01

IV. NUMERICAL EXPERIMENTS

In this section, we perform some numerical experiments to confirm the theoretical results in Section III. All numerical tests we perform in this section use the C++ library: `AFEPack`, the readers are referred to browse `http://WWW.acm.caltech.edu/ rli/AFEPack`.

Let $\Omega=(0,1)\times(0,1)$ in $\mathbb{R}^2$ and $K=\left\{\|v\|_{0,n_e;\Omega}\le 1\right\}$ where the even n_e will be determined later. We consider the following model problem

$$\begin{cases} \min\limits_{v} \quad \mathcal{J}(v,z)=\frac{1}{2}\int_\Omega (z-y_d)^2+\frac{1}{2}\int_\Omega v^2 \\ s.t. \begin{cases} Az &= f+v \quad in\ \Omega, \\ z &= 0\ on\ \partial\Omega, \end{cases} \\ \quad z\in K. \end{cases} \tag{15}$$

associated with the exact solution

$$\begin{aligned} p &= sin(\pi x_1)sin(\pi x_2) \\ y &= p/N_0 \\ u &= -p \\ f &= 2\pi^2 y-u \\ y_d &= y+\lambda y^{n_e-1}-2\pi^2 p \end{aligned} \tag{16}$$

where the number N_0 is given by

$$N_0=\left(\frac{n_e-1}{n_e-2}\times\frac{n_e-3}{n_e-4}\times\ldots\times\frac{1}{2}\right)^{\frac{2}{n_e}}$$

so that $\|y\|_{0,n_e;\Omega}=1$, and the positive number λ will be determined later.

Experiment

Here we solve model problem (15) associated with $n_e=10$ and $\lambda=30$. We computing and obtain the finite element solution on different finite element partitions of Ω. The numerical results are presented in Table I.

The convergence rates obtained from the above results are listed as Table IV. It is clear that data of both tables are consistent with our results in Theorem 1.

TABLE II
CONVERGENT RATES OF NUMERICAL SOLUTIONS

Mesh	$\|y-y_h\|_{0,2}$	$\lvert\lambda-\lambda_h\rvert$	$\|u-u_h\|_{0,2}$	$\|y-y_h\|_{1,2}$
1→ 2	1.9908	2.0188	1.9972	0.9992
2→ 3	1.9966	2.0028	1.9974	0.9992
3→ 4	1.9988	2.0003	1.9989	0.9997

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REFERENCES

[1] R. A. Adams, *Sobolev Spaces.* Academic Press, New York, 1975.
[2] S. C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods.* Springer, New York, 1994.
[3] E. Casas, *Control of an elliptic problem with pointwise state constraints.* SIAM J. Control Optim., 24(6), pp 1309-1322, 1986.
[4] E. Casas, *Error estimates for the numerical approximation of semilinear elliptic control problems with finitely many state constraints.* ESAIM Control,Optimisation and Calculus of Variations, 8(2002), pp 345-374.
[5] P. G. Ciarlet, *The Finite Element Methods for Elliptic Problems.* North-Holland Publishing Company, Amsterdam, 1978.
[6] F. H. Clarke, *Optimization and Nonsmooth Analysis.* John Wiley Sons, Amsterdam, 1983.
[7] F. H. Clarke, *A new approach to Lagange multipliers.* Math. Oper. Res., 1(1976), pp 165-174.
[8] K. Deckelnick and M. Hinze, *Convergence of a Finite Element Approximation to a State-constrained Elliptic Control Problem.* SIAM Journal on Numerical Analysis, Vol. 45, pp 1937-1953, 2007.
[9] Y. B. Deng, Z. Y. Liu, J. F. Wu and Y. H. Wu, *Topology optimization of steady Navier-Stokes flow with body force.* Comput. Methods Appl. Mech. Engrg, 255, pp 306-32, 2013.
[10] Y. B. Deng, Z. Y. Liu, P. Zhang, Y. S. Liu and Y. H. Wu, *Topology optimization of unsteady incompressible Navier-Stokes flows.* Journal of Computational Physics, 230, pp 6688-6708, 2011.
[11] W. B. Liu, D. P. Yang and L. Yuan, *Finite element approximation of optimal control problem with integral state constraint.* SIAM J. Numer. Anal., 48(3), pp 1163C1185, 2011.
[12] M. Lysaker, A. Lundervold and X. C. Tai, *Noise removal using fourth-order partial differential equation with applications to medical magnetic resonance images in space and time.* IEEE Transactions on Image Processing, Vol. 12, No.12, pp 1579-1590, Dec 2003.
[13] H. F. Niu, L. Yuan and D. P. Yang, *Finite element analysis of optimal control problem governed by stokes equations with L^2-norm state-constraints.* Journal of Computational Mathematics, 29(5), pp 589-604, 2011.
[14] H. F. Niu and D. P. Yang, *Adaptive finite element method for an optimal control problem of Stokes flow with L^2-norm state constraint.* International Journal for Numerical Methods in Fluids, 69(3), pp 534-549, 2012.
[15] L. Yuan, *Finite element approximations of optimal control problems with state con- straints.* PhD Thesis, School of Mathematics of Shandong University, 2009.
[16] L. Yuan and D. P. Yang, *A Posteriori Error Estimates of Control Problem of PDE with Integral Constraint for State.* J. Comput. Math. 27(4), pp 525-542, 2009.