Simplification of Boundary Conditions for a Rough Surface

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Abstract. With homogenization theory developing, solving some properties of a rough surface becomes more and more simple. What we studied was a two-dimensional perfectly conducting periodic rough surface with a thin cover layer. This allows for the development of an isovalent boundary condition for the effective fields at such a surface. It is shown that the coefficients in this isovalent boundary condition can be expressed as electric and magnetic polarizability densities. And at last we got the simplification of boundary problem for a rough surface.

1. Introduction

The researches about the influence of rough surface on the electromagnetic wave propagation have a long history. An effective research on equivalent electromagnetic parameters of periodic structures was derived from homogenization method in the 1970s [1-2]. And at present, it is one of the most effective methods of analyzing the rough surface [3]. By this way, we needn't think of the detailed structure of the fields. So far, the problem of electromagnetic wave propagation on a periodic distributions ideal conductor with a thin magnetic dielectric layer has not been solved. In this paper we will apply homogenization theory in simplifying the boundary problem for a rough surface.

2. The calculation of homogenization theory

Assume an ideal conductive rough surface, as shown in figure 1.An electromagnetic wave is incident onto the two-dimensional geometry which has a thin period magnetic dielectric layer on the surface.

Maxwell's equations in the structure of materials are expressed as

$$\nabla\times\mathbf{E}^W = -j\omega\mathbf{B}^W, \nabla\times\mathbf{H}^W = j\omega\mathbf{D}^W, \tag{1}$$

$$\mathbf{B}^W = \mu\mathbf{H}^W, \mathbf{D}^W = \varepsilon\mathbf{E}^W. \tag{2}$$

W is on behalf of the whole fields, which contain effective field and boundary layer field [4].

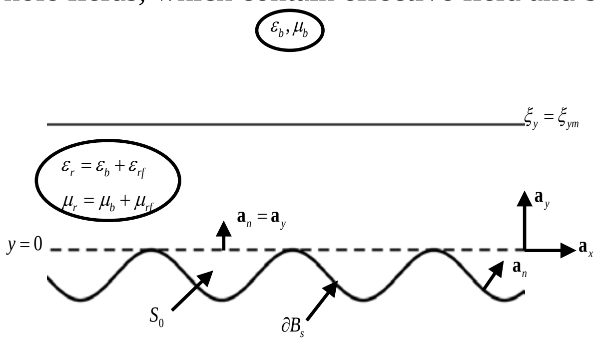

Fig. 1. Geometry of an ideal conductive rough surface

Due to the period length of the material p relative to macro scale of the whole structure is small. Macro-scale coordinates and micro-scale coordinates can be expressed respectively as follows [5].

$$\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z \tag{3}$$

$$\xi = \rho/p = \xi_x\mathbf{a}_x + \xi_y\mathbf{a}_y \tag{4}$$

Among them, $\rho = x\mathbf{a}_x + y\mathbf{a}_y$ is transverse macro-scale coordinates, the del operator ∇ is defined as

$$\nabla \to \nabla_{\mathbf{r}} + \frac{1}{p}\nabla_{\xi} . \tag{5}$$

Here $\nabla_{\mathbf{r}} = \mathrm{a}_x \frac{\partial}{\partial x} + \mathrm{a}_y \frac{\partial}{\partial y} + \mathrm{a}_z \frac{\partial}{\partial z}$, $\nabla_{\xi} = \mathrm{a}_x \frac{\partial}{\partial \xi_x} + \mathrm{a}_y \frac{\partial}{\partial \xi_y}$, so the $\mathbf{E}^W$ with multiple-scale index is $\mathbf{E}^W = \mathbf{E}^W(\mathbf{r}, \xi)$, the same as $\mathbf{H}^W$, $\mathbf{D}^W$, and $\mathbf{B}^W$.

$$\mathbf{E}^W = \mathbf{E}(\mathbf{r}) + \mathbf{e}(\mathbf{r}, \xi) \tag{6}$$

Assuming that $\mathbf{e}$, $\mathbf{h}$, $\mathbf{d}$ and $\mathbf{b}$ are periodic functions of ξ_x ,that is x/p ,and $\mathbf{e}, \mathbf{h}, \mathbf{b}, \mathbf{d} \to 0$ as $\xi_y \to \infty$.

With using (5), Formula (1) can be translated into

$$\nabla_r \times \mathbf{E} + \nabla_r \times \mathbf{e} + \frac{1}{p}\nabla_{\xi} \times \mathbf{e} = -j\omega(\mathbf{B} + \mathbf{b}), \nabla_r \times \mathbf{H} + \nabla_r \times \mathbf{h} + \frac{1}{p}\nabla_{\xi} \times \mathbf{h} = j\omega(\mathbf{D} + \mathbf{d}) \tag{7}$$

As shown in figure 1, assuming the space of relative dielectric constant and relative permeability can be expressed as

$$\varepsilon_r = \varepsilon_b + \varepsilon_{rf}(\xi_x, \xi_y), \mu_r = \mu_b + \mu_{rf}(\xi_x, \xi_y) \ , \tag{8}$$

and $\left.\begin{matrix}\varepsilon_{rf} \\ \mu_{rf}\end{matrix}\right\} \equiv 0 \qquad \xi_y > \xi_{ym}$, (9)

Here ε_b and μ_b are the material properties when there is cover layer. ε_{rf} and μ_{rf} are periodic distribution in ξ_x. $p\xi_y = y_m$ is is less than the cycle p. Equation (2) can be written as

$$\mathbf{B} = \mu_0 \mu_b \mathbf{H},\ \mathbf{D} = \varepsilon_0 \varepsilon_b \mathbf{E} . \tag{10}$$

Considering the effect of coupling of the boundary layer and the effective fields, (2) applied in the cover layer can be expressed as

$$\mathbf{b} = \mu_0 \left[\mu_r \mathbf{h} + \mu_{rf} \mathbf{H}\right], \mathbf{d} = \varepsilon_0 \left[\varepsilon_r \mathbf{e} + \varepsilon_{rf} \mathbf{E}\right] . \tag{11}$$

When $\xi_y \to \infty$, the influence of boundary layer can be neglected. So (7) can be represented as

$$\nabla_r \times \mathbf{E} = -jc\mathbf{B}, \nabla_r \times \mathbf{H} = jc\mathbf{D} \ . \tag{12}$$

Removes (12) from (7), we can gain

$$\nabla_r \times \mathbf{e} + \frac{1}{p}\nabla_{\xi} \times \mathbf{e} = -j\omega\mathbf{b}, \nabla_r \times \mathbf{h} + \frac{1}{p}\nabla_{\xi} \times \mathbf{h} = j\omega\mathbf{d} . \tag{13}$$

Since ξ_y of the boundary layer field can be ignored [6]. As a result, the boundary layer field is just a function of x, z, ξ_x, ξ_y, which is $\mathbf{e}(\mathbf{r}_0, \xi_x, \xi_y)$, among them $\mathbf{r}_0 = \mathbf{a}_x x + \mathbf{a}_z z$.

When the period length p is much smaller than a wavelength, the fields can be expanded approximation on p .

$$\mathbf{E} \sim \mathbf{E}^0(\mathbf{r}) + p\mathbf{E}^1(\mathbf{r}) + o(v^2), \mathbf{e} \sim \mathbf{e}^0(\mathbf{r}_0, \xi_x, \xi_y) + \mathrm{p}\mathbf{e}^1(\mathbf{r}_0, \xi_x, \xi_y) + o(v^2) \tag{14}$$

In order to calculate conveniently, the fields in the boundary layer can be seen as Taylor series expansion at $y = 0$. And any function of $\mathbf{r}$ is thus expanded in the boundary layer as

$$f(\mathbf{r}) = f(x, 0, z) + p\xi_y \frac{\partial f(x, y, z)}{\partial y}\Big|_{y=0} + o(v^2) \ . \tag{15}$$

Where $y = p\xi_y$ was used. Suppose $y = 0$ at the top of the rough surface, as shown in figure 1. Applying (15) to (11) , then substitute (14) into the results,

$$p^0 \quad \mathbf{b}^0 = \mu_0 \left[\mu_r \mathbf{h}^0 + \mu_{rf} \mathbf{H}^0(\mathbf{r}_0)\right], \mathbf{d}^0 = \varepsilon_0 \left[\varepsilon_r \mathbf{e}^0 + \varepsilon_{rf} \mathbf{E}^0(\mathbf{r}_0)\right] , \tag{16}$$

$$p^1 \quad \mathbf{b}^1 = \mu_0 \left[\mu_r \mathbf{h}^1 + \mu_{rf} \mathbf{H}^1(\mathbf{r}_0) + \mu_{rf} \xi_y \left(\frac{\partial \mathbf{H}^0}{\partial y} \right)_{y=0} \right], \mathbf{d}^1 = \varepsilon_0 \left[\varepsilon_r \mathbf{e}^1 + \varepsilon_{rf} \mathbf{E}^1(\mathbf{r}_0) + \varepsilon_{rf} \xi_y \left(\frac{\partial \mathbf{E}^0}{\partial y} \right)_{y=0} \right]. \quad (17)$$

Similarly, substitute (14) into (13), we can get

$$p^{-1} \quad \nabla_\xi \times \mathbf{e}^0 = 0, \nabla_\xi \times \mathbf{h}^0 = 0, \quad (18)$$

$$p^0 \quad \nabla_\xi \times \mathbf{e}^1 = -j\omega \mathbf{b}^0 - \nabla_r \times \mathbf{e}^0, \ \nabla_\xi \times \mathbf{h}^1 = j\omega \mathbf{d}^0 - \nabla_r \times \mathbf{h}^0, \quad (19)$$

$$p^1 \quad \nabla_\xi \times \mathbf{e}^2 = -j\omega \mathbf{b}^1 - \nabla_r \times \mathbf{e}^1, \nabla_\xi \times \mathbf{h}^2 = j\omega \mathbf{d}^1 - \nabla_r \times \mathbf{h}^1 . \quad (20)$$

From (19) and (20), we can obtain

$$\nabla_\xi \cdot \mathbf{b}^0 = 0, \nabla_\xi \cdot \mathbf{d}^0 = 0, \quad (21)$$

$$\nabla_\xi \cdot \mathbf{b}^1 = -\nabla_{\mathbf{r}} \cdot \mathbf{b}^0, \nabla_\xi \cdot \mathbf{d}^1 = -\nabla_{\mathbf{r}} \cdot \mathbf{d}^0 . \quad (22)$$

From these formulas, we can see the effective field depends on (12), and the boundary layer depends on the zero order electrostatic field (18) and (21), the first order electrostatic field (19) and (22).

3. The application of the calculation formulations

3.1 Boundary layer fields.

Assume the unit normal vector of the conductor boundary ∂B_s as shown in figure 1 is a_n. Then,

$$a_n \times \mathbf{E}^W \Big|_{\partial B_s} = 0 . \quad (23)$$

Choosing the plane $y = 0$ as effective field plane of isovalent boundary conditions, using the Taylor series expansion of (15) and (14), we can convert (23) into

$$p^0 \quad \mathbf{a}_n \times \mathbf{e}^0 = -\mathbf{a}_n \times \mathbf{E}^0(\mathbf{r}_0) \qquad \xi \in \partial B_s, \quad (24)$$

$$p^1 \quad \mathbf{a}_n \times \mathbf{e}^1 = -\xi_y \mathbf{a}_n \times \left[\left(\frac{\partial \mathbf{E}^0(\mathbf{r})}{\partial y} \right)_{y=0} \right] - \mathbf{a}_n \times \mathbf{E}^1(\mathbf{r}_0) \qquad \xi \in \partial B_s . \quad (25)$$

In the same way, using $a_n \cdot \mathbf{B}^W \Big|_{\partial B_s} = 0$, we obtain

$$p^0 \quad \mathbf{a}_n \cdot \mathbf{b}^0 = -\mathbf{a}_n \cdot \mathbf{B}^0(\mathbf{r}_0) \qquad \xi \in \partial B_s, \quad (26)$$

$$p^1 \quad \mathbf{a}_n \cdot \mathbf{b}^1 = -\xi_y \mathbf{a}_n \cdot \left[\left(\frac{\partial \mathbf{B}^0(\mathbf{r})}{\partial y} \right)_{y=0} \right] - \mathbf{a}_n \cdot \mathbf{B}^1(\mathbf{r}_0) \qquad \xi \in \partial B_s . \quad (27)$$

The fields are represented as (18), (21), (24) and (26). These formulas we can write again are

$$\nabla_\xi \times \mathbf{e}^0 = 0 \Rightarrow \xi \in \mathbf{B}, \nabla_\xi \cdot \left(\varepsilon_r \mathbf{e}^0 \right) = -\nabla_\xi \cdot \left(\varepsilon_r \mathbf{E}^0(\mathbf{r}_0) \right) \Rightarrow \xi \in \mathbf{B}, \mathbf{a}_n \times \mathbf{e}^0 = -\mathbf{a}_n \times \mathbf{E}^0(\mathbf{r}_0) \Rightarrow \xi \in \partial B_s, \quad (28)$$

$$\nabla_\xi \times \mathbf{h}^0 = 0 \Rightarrow \xi \in \mathbf{B}, \nabla_\xi \cdot \left(\mu_r \mathbf{h}^0 \right) = -\nabla_\xi \cdot \left(\mu_r \mathbf{H}^0(\mathbf{r}_0) \right) \Rightarrow \xi \in \mathbf{B}, \mathbf{a}_n \cdot \mathbf{b}^0 = -\mathbf{a}_n \cdot \mathbf{B}^0(\mathbf{r}_0) \Rightarrow \xi \in \partial B_s . \quad (29)$$

B is the region, As shown in figure 2.

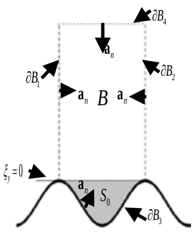

Fig. 2. Surface S_0 (shaded region) and B (shaded andunshaded regions)in a period cell, with boundaries.

The zeroth order boundary layer field is divided into two parts, such as

$$\mathbf{h}^0 = \mathbf{h}_t^{\,0} + \mathbf{a}_z h_z^{\,0}, \quad (30)$$

where $\mathbf{h}_t^{\,0} = \mathbf{a}_x h_x^{\,0} + \mathbf{a}_y h_y^{\,0}$.

So boundary conditions at the plane $y=0$ is

$$\mathbf{a}_y \times \mathbf{E}^0(\mathbf{r}_0)=0,\ \mathbf{a}_y \cdot \mathbf{B}^0(\mathbf{r}_0)=0 \ . \tag{31}$$

According to the last equations of (28), e^0 depend on ξ and can be written as

$$\mathbf{e}^0 = E_y{}^0(\mathbf{r}_0)\mathbf{A}(\xi_x,\xi_y)\ . \tag{32}$$

Here $\mathbf{A}(\xi_x,\xi_y)$ is independent on the effective field, and is perpendicular to z .

From the paper above, $\mathbf{h}^0$ can be written as

$$\mathbf{h}^0 = H_x{}^0(\mathbf{r}_0)\mathbf{C}(\xi_x,\xi_y)\ . \tag{33}$$

Here $\mathbf{C}(\xi_x,\xi_y)$ is also independent on the effective field, and is perpendicular to z .

For (16),

$$\mathbf{b}^0 = \mu_0\left[\mu_r\mathbf{h}^0 + \mu_{rf}\mathbf{H}^0(\mathbf{r}_0)\right] = \mu_0\left[H_x{}^0(\mathbf{r}_0)\left(\mu_r\mathbf{C}(\xi_x,\xi_y)+\mu_{rf}\mathbf{a}_x\right)+\mu_{rf}\mathbf{a}_z H_z{}^0(\mathbf{r}_0)\right] , \tag{34}$$

$$\mathbf{d}^0 = \varepsilon_0\left[\varepsilon_r\mathbf{e}^0 + \varepsilon_{rf}\mathbf{E}^0(\mathbf{r}_0)\right] = \varepsilon_0\left[E_y{}^0(\mathbf{r}_0)\left(\varepsilon_r\mathbf{A}(\xi_x,\xi_y)+\varepsilon_{rf}\mathbf{a}_y\right)\right]. \tag{35}$$

Make $\mathbf{M}(\xi_x,\xi_y)=\mu_r\mathbf{C}(\xi_x,\xi_y)+\mu_{rf}\mathbf{a}_x$, $\mathbf{N}(\xi_x,\xi_y)=\varepsilon_r\mathbf{A}(\xi_x,\xi_y)+\varepsilon_{rf}\mathbf{a}_y$, and equations (34) can be expressed as

$$\mathbf{b}^0 = \mu_0\left[H_x{}^0(\mathbf{r}_0)\mathbf{M}(\xi_x,\xi_y)+\mu_{rf}\mathbf{a}_z H_z{}^0(\mathbf{r}_0)\right],\ \mathbf{d}^0=\varepsilon_0 E_y{}^0(\mathbf{r}_0)\mathbf{N}(\xi_x,\xi_y)\ . \tag{36}$$

From (32), (33) and (36), we know the change of y has nothing to do with $\mathbf{e}^0$, $\mathbf{h}^0$, $\mathbf{b}^0$ and $\mathbf{d}^0$.

Now we use $\nabla_{\mathbf{r}}\times$ (32) and (33), respectively, with the properties of the boundary layer.

$$\nabla_{\mathbf{r}}\times\mathbf{e}^0 = -\mathbf{A}\times\nabla_{\mathbf{r}}E_y{}^0(\mathbf{r}_0),\ \nabla_{\mathbf{r}}\times\mathbf{h}^0 = -\mathbf{C}\times\nabla_{\mathbf{r}}H_x{}^0(\mathbf{r}_0)\ . \tag{37}$$

Using (37) and (36), (19) can be written as

$$\begin{aligned}
&\nabla_\xi\times\mathbf{e}^1 = \mathbf{A}\times\nabla_{\mathbf{r}}E_y{}^0(\mathbf{r}_0) - jk_0\eta_0\left[H_x{}^0(\mathbf{r}_0)\mathbf{M}(\xi_x,\xi_y)+\mu_{rf}\mathbf{a}_z H_z{}^0(\mathbf{r}_0)\right]\\
&\nabla_\xi\times\mathbf{h}^1 = \mathbf{C}\times\nabla_{\mathbf{r}}H_x{}^0(\mathbf{r}_0) + j\frac{k_0}{\eta_0}E_y{}^0(\mathbf{r}_0)\mathbf{N}(\xi_x,\xi_y)
\end{aligned} , \tag{38}$$

where $\eta_0=\sqrt{\dfrac{\mu_0}{\varepsilon_0}}$, $k_0=\omega\sqrt{\varepsilon_0\mu_0}$.

3.2 Simplification of boundary conditions.

With another kind of form of Gauss's theorem [7], the integral of $\nabla_\xi\times\mathbf{e}^1$ over the surface B can be expressed as

$$\int_B \nabla_\xi\times\mathbf{e}^1 dS_\xi = -\int_{\partial B}(\mathbf{a}_n\times\mathbf{e}^1)dl_\xi = -\int_{\partial B_3}(\mathbf{a}_n\times\mathbf{e}^1)dl_\xi\ . \tag{39}$$

Substituting the first equation of (38) into $\int_B \nabla_\xi\times\mathbf{e}^1 dS_\xi$, and substituting (25) into $-\int_{\partial B_3}(\mathbf{a}_n\times\mathbf{e}^1)dl_\xi$, we can obtain

$$\begin{aligned}
\mathbf{a}_y\times\mathbf{E}^1(\mathbf{r}_0) = {} & S_0\mathbf{a}_y\times\left[\left(\frac{\partial\mathbf{E}^0(\mathbf{r})}{\partial y}\right)_{y=0}\right] + \left(\int_B \mathbf{A}dS_\xi\right)\times\nabla_{\mathbf{r}}E_y{}^0(\mathbf{r}_0)\\
& - jk_0\eta_0 H_x{}^0(\mathbf{r}_0)\int_B \mathbf{M}dS_\xi - jk_0\eta_0\mathbf{a}_z H_z{}^0(\mathbf{r}_0)\int_B \mu_{rf}\,dS_\xi
\end{aligned} . \tag{40}$$

Since $\mathbf{E}(\mathbf{r}_0)\sim\mathbf{E}^0(\mathbf{r}_0)+p\mathbf{E}^1(\mathbf{r}_0)+o(v^2)$ and $\mathbf{a}_y\times\mathbf{E}^0(\mathbf{r}_0)=0$ of (31),

$$\mathbf{a}_y\times\mathbf{E}(\mathbf{r}_0) = \mathbf{a}_y\times\left[\mathbf{E}^0(\mathbf{r})+p\mathbf{E}^1(\mathbf{r})+o(v^2)\right] \approx p\mathbf{a}_y\times\mathbf{E}^1(\mathbf{r})\ . \tag{41}$$

From Maxwell's equations, we can get

$$\left(\frac{\partial\mathbf{E}^0(\mathbf{r})}{\partial y}\right)_{y=0} = \nabla_{\mathbf{r}}E_y{}^0(\mathbf{r}_0) + jk_0\eta_0\mu_b\left[\mathbf{a}_x H_z{}^0(\mathbf{r}_0) - \mathbf{a}_z H_x{}^0(\mathbf{r}_0)\right]. \tag{42}$$

With (40), (41), (42), the boundary condition of the effective field at $y=0$ can be written as

$$\begin{aligned}\mathbf{a}_y \times \mathbf{E}(\mathbf{r}_0) = p\Big[&\left(S_0\mathbf{a}_y + \int_B \mathbf{A} dS_\xi\right)\times\nabla_{\mathbf{r}} E_y^{\ 0}(\mathbf{r}_0) - jk_0\eta_0 H_x^{\ 0}(\mathbf{r}_0)\left(\int_B \mathbf{M} dS_\xi + \mu_b S_0 \mathbf{a}_x\right)\\ &- jk_0\eta_0 \mathbf{a}_z H_z^{\ 0}(\mathbf{r}_0)\left(\int_B \mu_{rf} dS_\xi + \mu_b S_0\right)\Big]\end{aligned}. \tag{43}$$

According to the conclusions, we get $\int_B \mathbf{A} dS_\xi = \mathbf{a}_y X(\frac{\varepsilon_r}{\varepsilon_b})$ and $\int_B \mathbf{M} dS_\xi = \mathbf{a}_x \mu_b X(\frac{\mu_b}{\mu_r})$ .Where X is a function of the normalization permittivity or permeability [8]. And we can get the surface electric polarizability density and magnetic polarizability density ρ_{eS} and ρ_{mS} .Let formulas $\rho_{mS,xx} = p\rho_{mx}$, $\rho_{mS,zz} = p\rho_{mz}$ and $\rho_{eS,yy} = p\rho_{ey}$ be established, we can gain

$$\rho_{mx} = -\left[X(\frac{\varepsilon_r}{\varepsilon_b}) + S_0\right],\ \rho_{mz} = -\left[X(\frac{\mu_b}{\mu_r}) + S_0\right],\ \rho_{ey} = -\left[\int_B \frac{\mu_{rf}}{\mu_b} dS_\xi + S_0\right]. \tag{44}$$

Through these polariability densities, simplification of boundary conditions for a periodic rough surface is finally expressed as

$$\begin{aligned}\mathbf{a}_y \times \mathbf{E}(\mathbf{r}_0) &= p\left[-\rho_{ey}\mathbf{a}_y \times \nabla_{\mathbf{r}} E_y^{\ 0}(\mathbf{r}_0) + jk_0\eta_0 H_x^{\ 0}(\mathbf{r}_0)\mu_b\rho_{mx}\mathbf{a}_x + jk_0\eta_0\mathbf{a}_z H_z^{\ 0}(\mathbf{r}_0)\mu_b\rho_{mz}\right]\\ &= j\omega p\left(\mathbf{a}_x B_x^{\ 0}(\mathbf{r}_0)\rho_{mx} + \mathbf{a}_z B_z^{\ 0}(\mathbf{r}_0)\rho_{mz}\right) - p\rho_{ey}\mathbf{a}_y \times \nabla_{\mathbf{r}} E_y^{\ 0}(\mathbf{r}_0)\end{aligned}. \tag{45}$$

We can see this equation is the same as the generalized impedance boundary conditions [9]. And its model had planar mental backed dielectric layers.

4. Summary

In this paper we have present the simplification of boundary conditions and it can be used to study scattering from periodic surfaces with a thin cover layer as the same way in the paper. Actually, the rough surface with a cover layer can be seen as an isovalent smooth surface which is located at the $y=0$ plane. The condition given in (45) is then applied at the smooth surface. All the effects of the model structure such as the roughness profile and the cover layer are equal to these boundary conditions. Of course, the equation has its own limitations. As the period length of the roughness and the thickness of the cover layer are smaller, the boundary-layer fields become more localized and its result is more accurate. For example, they will have no effect on the boundary condition of the effective fields and on the total fields and the case of a smooth uncoated perfect conductor is obtained.

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