Solution of a Class of High Order Differential Equations

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Abstract—In order to solve physical problems, we must establish mathematical models for the problems. Mathematical models often are differential equations relating an unknown function and one or more of its derivatives. In this paper, in order to solve high order differential equations, we first proved the expression for n repeated definite integrals by mathematical induction, Integration by parts and binomial formula. Then we obtained the solution of a class of high order differential equations by integral technique and the formula for n repeated definite integrals. Our results can be used to study the properties of high order differential equations, and then our results can be used to investigate physical or "real life" problems.

Keywords-Higher Order Differential Equation; Integration by Parts; Binomial Formula; Induction; Solution

I. INTRODUCTION

In order to solve physical problems, we must establish mathematical models for the problems. Mathematical models often are differential equations relating an unknown function and one or more of its derivatives. Wide applications of differential equations have attracted great interests of many mathematicians. Some mathematicians investigated existence of solutions for differential equations in [1-6]. Some mathematicians studied the solutions of high order differential equations and the theory on differential equations in [7-11].

II. MAIN RESULTS

In this section, we use the following lemmas to prove our main results.

Lemma 1. (see [12]) Suppose that $f(t)$ is a continuous function on $[a,\infty)$. Then the function

$$y(t) = \int_a^t \int_a^s \int_a^{\tau_1} \cdots \int_a^{\tau_{n-1}} f(\tau_n)(d\tau_n \cdots d\tau_2 d\tau_1 ds)$$
$$= \int_a^t \frac{(t-s)^n}{(n)!} f(s)ds, \qquad (1)$$

is a solution of higher order differential equation

$$\begin{cases} y^{(n+1)}(t) = f(t), t \in [a,\infty), \\ y^{(i)}(a) = 0, i = 0,1,\text{L},n. \end{cases} \qquad (2)$$

Proof. For the reader's convenience, we prove this Lemma by induction. Firstly, we consider the case of $n = 1$. The solution of differential equation (2) is

$$y(t) = \int_a^t \int_a^s f(\tau_1)d\tau_1 ds$$
$$= s\int_a^s f(\tau_1)d\tau_1 \mid_a^t - \int_a^t sf(s)ds$$
$$= \int_a^t f(t-s)f(s)ds. \qquad (3)$$

So (1) is true for $n = 1$. Letting this be our basis step, suppose that (1) is true up to some $k > 1$, i.e. the solution of differential equation (2) is

$$y(t) = \int_a^t \int_a^s \int_a^{\tau_1} \cdots \int_a^{\tau_{k-1}} f(\tau_k)(d\tau_k \cdots d\tau_2 d\tau_1 ds)$$
$$= \int_a^t \frac{(t-s)^k}{(k)!} f(s)ds. \qquad (4)$$

If $n = k+1$, then the induction hypothesis implies that the solution of differential equation (2) is

$$y(t) = \int_a^t \int_a^s \int_a^{\tau_1} \cdots \int_a^{\tau_k} f(\tau_{k+1})(d\tau_{k+1} \cdots d\tau_2 d\tau_1 ds)$$
$$= \int_a^t \int_a^s \frac{(s-\tau_1)^k}{(k)!} f(\tau_1)d\tau_1 ds.$$
$$= s\int_a^s \frac{(s-\tau_1)^k}{(k)!} f(\tau_1)d\tau_1 \mid_a^t$$
$$- \int_a^t s\int_a^s \frac{(s-\tau_1)^{k-1}}{(k-1)!} f(\tau_1)d\tau_1 ds$$
$$= t\int_a^t \frac{(t-s)^k}{(k)!} f(s)ds - (\frac{s^2}{2}\int_a^s \frac{(s-\tau_1)^{k-1}}{(k-1)!} f(\tau_1)d\tau_1 \mid_a^t$$
$$- \int_a^t \frac{s^2}{2}\int_a^s \frac{(s-\tau_1)^{k-2}}{(k-2)!} f(\tau_1)d\tau_1 ds)$$
$$= t\int_a^t \frac{(t-s)^k}{(k)!} f(s)ds - (\frac{t^2}{2}\int_a^t \frac{(t-s)^{k-1}}{(k-1)!} f(s)ds$$
$$- \int_a^t \frac{s^2}{2}\int_a^s \frac{(s-\tau_1)^{k-2}}{(k-2)!} f(\tau_1)d\tau_1 ds). \qquad (5)$$

We consider the integration

$$\int_a^t \frac{s^2}{2} \int_a^s \frac{(s-\tau_1)^{k-2}}{(k-2)!} f(\tau_1)\,d\tau_1\,ds. \qquad (6)$$

When $k-2=1$, we have

$$\int_a^t \frac{s^2}{2} \int_a^s (s-\tau_1) f(\tau_1)\,d\tau_1\,ds$$

$$= \frac{s^3}{3!} \int_a^s (s-\tau_1) f(\tau_1)\,d\tau_1 \Big|_a^t - \int_a^t \frac{s^3}{3!} \int_a^s f(\tau_1)\,d\tau_1\,ds$$

$$= \frac{t^3}{3!} \int_a^t (t-s) f(s)\,ds$$

$$- \left(\frac{s^4}{4!} \int_a^s f(\tau_1)\,d\tau_1 \Big|_a^t - \int_a^t \frac{s^4}{4!} f(s)\,ds\right)$$

$$= \frac{t^3}{3!} \int_a^t (t-s) f(s)\,ds - \frac{t^4}{4!} \int_a^t f(s)\,ds + \int_a^t \frac{s^4}{4!} f(s)\,ds$$

$$= \int_a^t \frac{\dfrac{4!}{3!1!} t^3 (t-s) - t^4 + s^4}{4!} f(s)\,ds. \qquad (7)$$

When $k-2=2$, we have

$$\int_a^t \frac{s^2}{2} \int_a^s \frac{(s-\tau_1)^2}{2!} f(\tau_1)\,d\tau_1\,ds$$

$$= \frac{s^3}{3!} \int_a^s \frac{(s-\tau_1)^2}{2!} f(\tau_1)\,d\tau_1 \Big|_a^t$$

$$- \int_a^t \frac{s^3}{3!} \int_a^s (s-\tau_1) f(\tau_1)\,d\tau_1\,ds$$

$$= \frac{t^3}{3!} \int_a^t \frac{(t-s)^2}{2!} f(s)\,ds$$

$$- [\frac{s^4}{4!} \int_a^s (s-\tau_1) f(\tau_1)\,d\tau_1 \Big|_a^t - \int_a^t \frac{s^4}{4!} \int_a^s f(\tau_1)\,d\tau_1\,ds]$$

$$= \frac{t^3}{3!} \int_a^t \frac{(t-s)^2}{2!} f(s)\,ds - \{\frac{s^4}{4!} \int_a^s (s-\tau_1) f(\tau_1)\,d\tau_1 \Big|_a^t$$

$$- [\frac{s^5}{5!} \int_a^s f(\tau_1)\,d\tau_1 \Big|_a^t - \int_a^t \frac{s^5}{5!} f(s)\,ds]\}$$

$$= \frac{t^3}{3!} \int_a^t \frac{(t-s)^2}{2!} f(s)\,ds - \{\frac{t^4}{4!} \int_a^t (t-s) f(s)\,ds$$

$$- [\frac{t^5}{5!} \int_a^t f(s)\,ds - \int_a^t \frac{s^5}{5!} f(s)\,ds]\}$$

$$= \int_a^t \frac{\dfrac{5!}{3!2!} t^3 (t-s)^2 - \dfrac{5!}{4!1!} t^4 (t-s) + t^5 - s^5}{5!} f(s)\,ds. \qquad (8)$$

When $k-2=3$, we get

$$\int_a^t \frac{s^2}{2} \int_a^s \frac{(s-\tau_1)^3}{3!} f(\tau_1)\,d\tau_1\,ds$$

$$= \frac{s^3}{3!} \int_a^s \frac{(s-\tau_1)^3}{3!} f(\tau_1)\,d\tau_1 \Big|_a^t$$

$$- \int_a^t \frac{s^3}{3!} \int_a^s \frac{(s-\tau_1)^2}{2!} f(\tau_1)\,d\tau_1\,ds$$

$$= \frac{t^3}{3!} \int_a^t \frac{(t-s)^3}{3!} f(s)\,ds$$

$$- [\frac{s^4}{4!} \int_a^s \frac{(s-\tau_1)^2}{2!} f(\tau_1)\,d\tau_1 \Big|_a^t$$

$$- \int_a^t \frac{s^4}{4!} \int_a^s (s-\tau_1) f(\tau_1)\,d\tau_1\,ds]$$

$$= \frac{t^3}{3!} \int_a^t \frac{(t-s)^3}{3!} f(s)\,ds - \{\frac{s^4}{4!} \int_a^s \frac{(s-\tau_1)^2}{2!} f(\tau_1)\,d\tau_1 \Big|_a^t$$

$$- [\frac{s^5}{5!} \int_a^s (s-\tau_1) f(\tau_1)\,d\tau_1 \Big|_a^t - \int_a^t \frac{s^5}{5!} \int_a^s f(\tau_1)\,d\tau_1\,ds]\}$$

$$= \frac{t^3}{3!} \int_a^t \frac{(t-s)^3}{3!} f(s)\,ds - \{\frac{t^4}{4!} \int_a^t \frac{(t-s)^2}{2!} f(s)\,ds$$

$$- [\frac{t^5}{5!} \int_a^t (t-s) f(\tau_1)\,ds - (\frac{t^6}{6!} \int_a^t f(s)\,ds - \int_a^t \frac{s^6}{6!} f(s)\,ds)]\}$$

$$= \frac{1}{6!} \int_a^t (\frac{6!}{3!3!} t^3 (t-s)^3 - \frac{6!}{4!2!} t^4 (t-s)^2) f(s)\,ds.$$

$$+ \frac{1}{6!} \int_a^t (\frac{6!}{5!1!} t^5 (t-s) - t^6 + s^6) f(s)\,ds. \qquad (9)$$

By the same procedure as above, we get

$$\int_a^t \frac{s^2}{2} \int_a^s \frac{(s-\tau_1)^{m-2}}{(m-2)!} f(\tau_1)\,d\tau_1\,ds$$

$$= \frac{t^3}{3!} \int_a^t \frac{(t-s)^{m-2}}{(m-2)!} f(s)\,ds$$

$$+ (-1)^3 \frac{t^4}{4!} \int_a^t \frac{(t-s)^{m-3}}{(m-3)!} f(s)\,ds + \cdots$$

$$+ (-1)^{m-2} \frac{t^{m-1}}{(m-1)!} \int_a^t \frac{(t-s)^2}{2!} f(s)\,ds$$

$$+ (-1)^{m-1} \frac{t^m}{m!} \int_a^t (t-s) f(s)\,ds$$

$$+ (-1)^m \frac{t^{m+1}}{(m+1)!} \int_a^t f(s)\,ds + (-1)^{m+1} \int_a^t \frac{s^{m+1}}{(m+1)!} f(s)\,ds$$

$$= \frac{1}{(m+1)!} \int_a^t [\frac{(m+1)!}{3!(m-2)!} t^3 (t-s)^{m-2}$$

$$- \frac{(m+1)!}{4!(m-3)!} t^4 (t-s)^{m-3} + \cdots$$

$$+(-1)^{m-2}\frac{(m+1)!}{(m-1)!2!}t^{m-1}(t-s)^2$$

$$+(-1)^{m-1}\frac{(m+1)!}{m!1!}t^m(t-s)+(-1)^m t^{m+1}$$

$$+(-1)^{m+1}s^{m+1}]f(s)ds.$$

Taking into account (5) and (10), we obtain

$$y(t)=\int_a^t\int_a^s\int_a^{\tau_1}\cdots\int_a^{\tau_k}f(\tau_{k+1})(d\tau_{k+1}\cdots d\tau_2 d\tau_1 ds)$$

$$=t\int_a^t\frac{(t-s)^k}{k!}f(s)ds-(\frac{t^2}{2}\int_a^t\frac{(t-s)^{k-1}}{(k-1)!}f(s)ds$$

$$-\int_a^t\frac{s^2}{2}\int_a^s\frac{(s-\tau_1)^{k-2}}{(k-2)!}f(\tau_1)d\tau_1 ds)$$

$$=t\int_a^t\frac{(t-s)^k}{k!}f(s)ds-(\frac{t^2}{2}\int_a^t\frac{(t-s)^{k-1}}{(k-1)!}f(s)ds$$

$$-\frac{1}{(k+1)!}\int_a^t[\frac{(k+1)!}{3!(k-2)!}t^3(t-s)^{k-2}-\frac{(k+1)!}{4!(k-3)!}t^4(t-s)^{k-3}$$

$$+\cdots+(-1)^{k-2}\frac{(k+1)!}{(k-1)!2!}t^{k-1}(t-s)^2$$

$$+(-1)^{k-1}(k+1)t^k(t-s)+(-1)^k t^{k+1}$$

$$+(-1)^k t^{k+1}+(-1)^{k+1}s^{k+1}]f(s)ds)$$

$$=\frac{1}{(k+1)!}\int_a^t[\frac{(k+1)!}{1!k!}t(t-s)^k-\frac{(k+1)!}{2!(k-1)!}t^2(t-s)^{k-1}$$

$$+\cdots+(-1)^{k-2}\frac{(k+1)!}{(k-1)!2!}t^{k-1}(t-s)^2$$

$$+(-1)^{k-1}(k+1)t^k(t-s)$$

$$+(-1)^k t^{k+1}+(-1)^{k+1}s^{k+1}]f(s)ds$$

$$=\frac{1}{(k+1)!}\int_a^t[-(t-s)^{k+1}+\frac{(k+1)!}{1!k!}t(t-s)^k$$

$$-\frac{(k+1)!}{2!(k-1)!}t^2(t-s)^{k-1}$$

$$+\cdots+(-1)^{k-2}\frac{(k+1)!}{(k-1)!2!}t^{k-1}(t-s)^2$$

$$+(-1)^{k-1}(k+1)t^k(t-s)+(-1)^k t^{k+1}$$

$$+(t-s)^{k+1}+(-1)^{k+1}s^{k+1}]f(s)ds)$$

$$=\int_a^t[\frac{-(-s)^{k+1}+(t-s)^{k+1}+(-1)^{k+1}s^{k+1}}{(k+1)!}]f(s)ds$$

$$=\int_a^t\frac{(t-s)^{k+1}}{(k+1)!}f(s)ds. \qquad (11)$$

Therefore, by induction, (17) is true for all $n>1$.

Theorem 1. Suppose that $f(t)$ is a continuous function on $[a,\infty)$, and $p(t)>0$ is an nth continuous differentiable function on $[a,\infty)$. Then the function

$$y(t)=\int_a^t\frac{(t-s)^{m-1}}{(m-1)!}(\frac{1}{p(s)}\int_a^s\frac{(s-\tau)^{n-1}}{(n-1)!}f(\tau)d\tau)ds \qquad (12)$$

is a solutions of higher order differential equation

$$\begin{cases}\dfrac{d^n}{dt^n}(p(t)\dfrac{d^m}{dt^m}y(t))=f(t),t\in[a,\infty),\\ y^{(i)}(a)=0,i=0,1,\cdots,n+m-1.\end{cases} \qquad (13)$$

Proof. Carrying out integrations on both sides of (13) n times, using (1), we obtain

$$p(t)\frac{d^m}{dt^m}y(t)=\int_a^t\int_a^s\int_a^{\tau_1}\mathsf{L}\int_a^{\tau_{n-2}}f(\tau_{n-1})(d\tau_{n-1}\mathsf{L}d\tau_2 d\tau_1 ds)$$

$$=\int_a^t\frac{(t-s)^{n-1}}{(n-1)!}f(s)ds.$$

Dividing both sides of (14) by $p(t)$, we have

$$\frac{d^m}{dt^m}y(t)=\frac{1}{p(t)}\int_a^t\frac{(t-s)^{n-1}}{(n-1)!}f(s)ds. \qquad (15)$$

From (15), we get

$$y(t)=\int_a^t\frac{(t-s)^{m-1}}{(m-1)!}(\frac{1}{p(s)}\int_a^s\frac{(s-\tau)^{n-1}}{(n-1)!}f(\tau)d\tau)ds. \qquad (16)$$

(16) is a solution of higher order differential equation (13).

Theorem 2. Suppose that c_1,c_2 are positive constants. Then the function

$$y(t)=\frac{c_1(t-a)^{m+n}}{c_2(m+n)!} \qquad (17)$$

is a solutions of higher order differential equation

$$\begin{cases}\dfrac{d^n}{dt^n}(c_2\dfrac{d^m}{dt^m}y(t))=c_1,t\in[a,\infty)\\ y^{(i)}(a)=0,i=0,1,\cdots,n+m-1.\end{cases} \qquad (18)$$

Proof. By Theorem 1, we see that the function

$$y(t)=\frac{c_1}{c_2}\int_a^t\frac{(t-s)^{m-1}}{(m-1)!}(\int_a^s\frac{(s-\tau)^{n-1}}{(n-1)!}d\tau)ds \qquad (19)$$

is a solutions of higher order differential equation

$$\begin{cases}\dfrac{d^n}{dt^n}(c_2\dfrac{d^m}{dt^m}y(t))=c_1,t\in[a,\infty),\\ y^{(i)}(a)=0,i=0,1,\cdots,n+m-1.\end{cases} \qquad (20)$$

From (14), we have

$$y(t)=\frac{c_1}{c_2}\int_a^t\frac{(t-s)^{m-1}}{(m-1)!}(\int_a^s\frac{(s-\tau)^{n-1}}{(n-1)!}d\tau)ds$$

$$=\frac{c_1}{c_2}\int_a^t\frac{(t-s)^{m-1}}{(m-1)!}(-\frac{(s-\tau)^n}{n!}|_a^s)ds$$

$$=\frac{c_1}{c_2}\int_a^t\frac{(t-s)^{m-1}(s-a)^n}{(m-1)!n!}ds$$

$$= \frac{c_1}{c_2} \int_a^t \frac{(t-s)^{m-1}(s-a)^n}{(m-1)!n!} ds . \qquad (21)$$

Let $s-a = x(t-a)$, then $t-s = t-a-(s-a)$. From (21), we have

$$y(t) = \frac{c_1(t-a)^{m+n}}{c_2} \int_0^1 \frac{(1-v)^{m-1}v^n}{(m-1)!n!} dv$$

$$= \frac{c_1(t-a)^{m+n}}{(m-1)1n!c_2} B(m, n+1)$$

$$= \frac{c_1(t-a)^{m+n} n!(m-1)!}{(m-1)!n!(m+n)!c_2}$$

$$= \frac{c_1(t-a)^{m+n}}{c_2(m+n)!} .$$

III.   CONCLUSIONS

In this paper, firstly, we proved the function

$$y(t) = \int_a^t \int_a^s \int_a^{\tau_1} L \int_a^{\tau_{n-1}} f(\tau_n)(d\tau_n L d\tau_2 d\tau_1 ds)$$

$$= \int_a^t \frac{(t-s)^n}{(n)!} f(s)ds$$

be a solution of higher order differential equation

$$\begin{cases} y^{(n+1)}(t) = f(t), t \in [a, \infty), \\ y^{(i)}(a) = 0, i = 0, 1, L, n. \end{cases}$$

Then we proved the function

$$y(t) = \int_a^t \frac{(t-s)^{m-1}}{(m-1)!} \left(\frac{1}{p(s)} \int_a^s \frac{(s-\tau)^{n-1}}{(n-1)!} f(\tau)d\tau \right) ds$$

be a solutions of higher order differential equation

$$\begin{cases} \frac{d^n}{dt^n} (p(t) \frac{d^m}{dt^m} y(t)) = f(t), t \in [a, \infty), \\ y^{(i)}(a) = 0, i = 0, 1, \cdots, n+m-1. \end{cases}$$

Finally, we proved the function

$$y(t) = \frac{c_1(t-a)^{m+n}}{c_2(m+n)!}$$

to be a solutions of higher order differential equation

$$\begin{cases} \frac{d^n}{dt^n} (c_2 \frac{d^m}{dt^m} y(t)) = c_1, t \in [a, \infty) \\ y^{(i)}(a) = 0, i = 0, 1, \cdots, n+m-1. \end{cases}$$

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