

Wave Analysis of a Diffusive Modified Leslie-Gower Predator-prey System with Holling Type IV Schemes

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Abstract. A diffusive predator-prey model with modified Leslie-Gower and Holling type IV schemes is investigated analytically and numerically. Mathematical theoretical works mainly focus on the existence of traveling wave solutions. Numerical simulations are performed to confirm the feasibility of traveling wave solutions. All these results are significant in exploration of the dynamic complexity of ecosystems.

Introduction

Predator-prey interactions have attracted considerable attention in mathematical biology. In the classical Lotka-Volterra predator-prey models, the growth rate of population is assumed to be proportional to its size. However, Leslie and Gower [1,2] initiated a predator-prey model where the carrying capacity of predator is proportional to the number of prey. Recently, the non-monotonic Holling type IV functional response has been widely used to describe the process of predation with self-selection and the inhibitory effect of prey [3,4,5]. On the other hand, all organisms interact with each other in a limited space. Meanwhile, the spatial value has been regarded as a pivotal role on how ecological communities are shaped [6,7,8]. From above, we can study the following equations:

$$\begin{cases} \frac{\partial H}{\partial T} = d_1 \Delta H + H(a_1 - b_1 H) - \frac{c_1 HP}{e_1 + H^2}, \\ \frac{dP}{dT} = d_2 \Delta P + P(a_2 - \frac{c_2 P}{e_2 + H}), \end{cases} \quad (1)$$

where $H(T, X)$ and $P(T, X)$ denote the densities of prey and predator at time T and position X , a_1 and a_2 correspond to the growth rates of prey and predator, b_1 reflects the competitive strength among individuals of prey, c_1 is the maximum value of the per capita reduction of H due to P , c_2/a_2 measures the ration of prey to support one predator, e_1 is interpreted as the half-saturation constant, e_2 indicates the quality of the alternative that provides the environment, Δ is the Laplacian operator, d_1 and d_2 are the diffusion coefficients of prey and predator. All the parameters of Eq. 1 are supposed to be positive. For simplicity, the scaling transformation is performed: $t = a_1 T$,

$$x = \left(\frac{a_1}{d_1}\right)^{\frac{1}{2}}, \quad u(t, x) = \frac{b_1}{a_1} H(T, X), \quad w(t, x) = \frac{b_1 c_2}{a_1 a_2} P(T, X), \quad s = \frac{a_2}{a_1}, \quad \delta = \frac{d_2}{d_1},$$

$$a = \frac{e_1 b_1^2}{a_1^2}, \quad b = \frac{b_1 e_2}{a_1}, \quad \alpha = \frac{c_1 c_2 b_1^3}{a_2 a_1^3}.$$

Assume that the interaction between prey and predator is limited by the bounded region Ω in R^n with the smooth boundary. This leads to the following spatial-temporal equations:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u(1-u) - \frac{\alpha u w}{a+u^2}, & x \in \Omega, t > 0, \\ \frac{\partial w}{\partial t} = \delta \Delta w + s w \left(1 - \frac{w}{b+u}\right), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \quad (2)$$

where $\partial/\partial n$ is differential in the direction of the outward unit normal to $\partial\Omega$. Clearly, Eq. 2 has the following equilibrium points:

$$E_0 = (0, 0), \quad E_1 = (1, 0), \quad E_2 = (0, b), \quad E_3 = (u_*, w_*) = (b + u_*, w_*),$$

where w_* is the positive root of $u^3 - u^2 + (a + \alpha)u + \alpha b - a = 0$.

Existence of Traveling Wave Solution

In this section, we establish the existence of traveling wave connecting E_1 and E_3 in Eq. 2. Assume that Eq. 2 has a solution in the form of $(h(x-ct), q(x-ct))$ such that $(h(-\infty), q(-\infty)) = E_3$ and $(h(+\infty), q(+\infty)) = E_1$, where c is the wave speed. Denote $x-ct$ by ξ and substitute $(u(t, x), w(t, x)) = (h(\xi), q(\xi))$ into Eq. 2. Thus, Eq. 2 has a pair of traveling wave solutions if and only if it has the following wave equations:

$$\begin{cases} h'' + ch' + hp_1(h, q) = 0, \\ \delta q'' + cq' + qp_2(h, q) = 0, \end{cases} \quad (3)$$

where $h' = \frac{dh}{d\xi}$, $h'' = \frac{d^2h}{d\xi^2}$, $q' = \frac{dq}{d\xi}$, $q'' = \frac{d^2q}{d\xi^2}$, $p_1(h, q) = 1 - h - \frac{\alpha q}{a + h^2}$,

$$p_2(h, q) = s \left(1 - \frac{q}{b+h}\right).$$

Assume that

$$S = [0, 1] \times [0, b+1] \text{ and } S_0 = \{(u, w) \in S, p_1(u, w) < 0, p_2(u, w) > 0, w \leq w_*\}.$$

Consider the differential equation

$$d \frac{d^2\varphi}{d\xi^2} + c \frac{d\varphi}{d\xi} + m\varphi = 0. \quad (4)$$

Suppose that

$$c^2 > 4dm \quad (5)$$

with $d = \max(1, \delta)$ and $m = \max_{(u, w) \in S} (p_1(u, w), p_2(u, w))$.

Lemma 1. ([9]) Assume that $\varphi: R_+ \rightarrow R$ is uniformly continuous and Riemann integrable, then $\lim_{\xi \rightarrow \infty} \varphi(\xi) = 0$.

Lemma 2. ([10]) Assume that d , c and m are positive constants such that $c^2 > 4dm$, then for any $\xi > \xi_0$, $\varphi(\xi) \neq 0$ provided that φ is a solution of Eq. 4 and $\varphi'(\xi_0) = 0$.

Lemma 3. Suppose that $(h(\xi), q(\xi))$ is a solution of Eq. 3 satisfying $(h(0), p(0)) \in S_0$ and $(h'(0), q'(0)) = (0, 0)$. If Eq. 5 holds, it follows that

(a) For any $\xi > 0$, $h' > 0$, $q' < 0$, $u_* < h < 1$, $0 < q < w_*$.

(b) $\lim_{\xi \rightarrow +\infty} h(\xi) = 1$, $\lim_{\xi \rightarrow +\infty} q(\xi) = 0$.

(c) $h'(\xi)$ and $q'(\xi)$ are uniformly bounded.

Proof. It is clear that $h(\xi) \geq 0$, $q(\xi) \geq 0$ for any $\xi > 0$. Note that $h'' = -hp_1(h, q) > 0$ and $q'' = -\frac{q}{\delta} p_2(h, q) > 0$ at $\xi = 0$. Thus, there exists an enough small neighborhood of $\xi = 0$ such that $h''(\xi) > 0$, $q''(\xi) > 0$, and $h'(\xi) > 0$, $q'(\xi) > 0$. Suppose that there exists $\bar{\xi} > 0$ such that $h'(\xi) > 0$ on $[0, \bar{\xi})$ and $h'(\bar{\xi}) = 0$. Let $\bar{\xi}$ denote the first such point and $\varphi(\xi) = \lambda_2 e^{\lambda_1 \xi} - \lambda_1 e^{\lambda_2 \xi}$ be the solution of

$$\varphi'' + c\varphi' + m\varphi = 0. \quad (6)$$

Then, there exist $\varphi(0) = \lambda_2 - \lambda_1$, $\varphi'(0) = 0$, where λ_2 and λ_1 are the roots of (6) and condition Eq. 5 is satisfied. Define $\rho(\xi) = e^{c\xi}$. By multiplying Eq. 6 by ρh and subtracting the product of the first equation of Eq. 3 with $\rho\varphi$, it has

$$\frac{d(\rho(h\varphi' - h'\varphi))}{d\xi} + (m - p_1(h, q)) = 0. \quad (7)$$

Integrating Eq. 7 from 0 to $\bar{\xi}$, there exist

$$\rho(\bar{\xi})h(\bar{\xi})\varphi'(\bar{\xi}) + \int_0^{\bar{\xi}} (m - p_1(h, q))\rho(\xi)h(\xi)d\xi = 0 \text{ and } (m - p_1(h, q)) > 0.$$

It leads to a contradiction since $\rho(\bar{\xi}) > 0$, $h(\bar{\xi}) > 0$ and $\varphi'(\bar{\xi}) > 0$. Thus, $h'(\xi) > 0$ for any $\xi > 0$. By applying the same method, it is clear that $q'(\xi) < 0$. Furthermore, it follows that $u_* \leq h(\xi) \leq 1$ and $0 \leq q(\xi) \leq w_*$ if $u_* \leq h(0) \leq 1$ and $0 \leq q(0) \leq w_*$. As a result, there exist \hat{h} and \hat{q} such that $\lim_{\xi \rightarrow +\infty} h(\xi) = \hat{h}$, $\lim_{\xi \rightarrow +\infty} q(\xi) = \hat{q}$. Meanwhile, from

Eq. 3, it has $h'' + ch' \leq \frac{\alpha h q}{h^2 + a}$, $\delta q'' + cq' \leq \frac{sq^2}{h + b}$. Since $h(\xi)$ and $q(\xi)$ are uniformly

bounded, it is clear that there exist two positive constants \hat{H} , \hat{Q} such that

$$h'' + ch' \leq \hat{H}, \quad \delta q'' + cq' \leq \hat{Q}. \quad (8)$$

From Eq.8, we obtain the uniform bound of $h'(\xi)$ and $q'(\xi)$. Consequently, $p''(\xi)$ and $q''(\xi)$ are also uniformly bounded. Applying Lemma Eq. 3 to $(h'(\xi), q'(\xi))$ and $(h''(\xi), q''(\xi))$, it follows that $\lim_{\xi \rightarrow +\infty} h'(\xi) = 0$, $\lim_{\xi \rightarrow +\infty} h''(\xi) = 0$ and $\lim_{\xi \rightarrow +\infty} q'(\xi) = 0$, $\lim_{\xi \rightarrow +\infty} q''(\xi) = 0$. Taking into account $\xi \rightarrow +\infty$ in Eq. 3, it has $f(\hat{H}, \hat{Q}) = 0$, $g(\hat{H}, \hat{Q}) = 0$, as a sequence of which it leads to $\hat{H} = 1$, $\hat{Q} = 0$. The above results show that any solution which starts at $\xi = 0$ in S_0 approaches to the boundary equilibrium E_1 as $t \rightarrow +\infty$.

Lemma 4. ([10]) Assume that there exist $\tau_{n,1} > 0$, $\tau_{n,2} > 0$ such that $(h_n(\tau_{n,1}), q_n(\tau_{n,2})) = (\frac{u_*}{2}, \frac{w_*}{2})$, then $\lim_{n \rightarrow +\infty} \tau_{n,i} = +\infty, i = 1, 2$.

Theorem 1. Assume that Eq. 6 is satisfied, then there exists a solution $(h(\xi), q(\xi))$ of Eq. 3 defined in R such that $(h(-\infty), q(-\infty)) = E_*$, $(h(+\infty), q(+\infty)) = E_1$ and $u_* < h < 1$, $0 < q < w_*$, $h' > 0, q' < 0$.

Proof. For any $n \in \mathbb{N}$, let $(h_n(\xi), q_n(\xi))$ be the solution of Eq. 3 with the initial conditions, $(h_n(0), q_n(0)) = (c_{n,1}, c_{n,2}) \in S_0$ and $(h'_n(0), q'_n(0)) = (0, 0)$, where $(c_{n,1}, c_{n,2}) \rightarrow (u_*, w_*)$ as $n \rightarrow +\infty$. From Lemma 2.3, for any $n \in \mathbb{N}$ and $\xi > 0$, it has $\frac{dh_n}{d\xi}(\xi) > 0$, $\frac{dq_n}{d\xi}(\xi) < 0$ and $\lim_{\xi \rightarrow +\infty} h_n(\xi) = 1$, $\lim_{\xi \rightarrow +\infty} q_n(\xi) = 0$. Thus, it needs to make further step of the traveling wave by considering a sequence of functions which converges into E_3 . For any $n \in \mathbb{N}$ and $\xi \geq \tau_{n,i}, i = 1, 2$, define $(\phi_{n,1}(\xi), \phi_{n,2}(\xi)) = (h_n(\xi + \tau_{n,1}), q_n(\xi + \tau_{n,2}))$ for $\xi \geq -\tau_{n,i}, i = 1, 2$. Since $(h'_n(\xi), q'_n(\xi))$ is uniformly bounded, then $(\phi'_{n,1}(0), \phi'_{n,2}(0))$ is bounded. Thus there is a sequence $(\phi_{n_k,1}(\xi), \phi_{n_k,2}(\xi))$ such that $(\phi'_{n_k,1}(0), \phi'_{n_k,2}(0)) = (h'_n(\tau_{n_k,1}), q'_n(\tau_{n_k,2})) \rightarrow (\theta_1, \theta_2)$ as $k \rightarrow +\infty$ with $\theta_1 \geq 0$ and $\theta_2 \leq 0$. Assume that $v = (v_1, v_2)$ is the solution of

$$\begin{cases} v_1'' + cv_1' + v_1 p_1(v_1, v_2) = 0, \\ \delta v_2'' + cv_2' + v_2 p_2(v_1, v_2) = 0, \end{cases} \quad (9)$$

satisfying $v(0) = (\frac{u_*}{2}, \frac{w_*}{2})$ and $v'(0) = (\theta_1, \theta_2)$. Then $\{\phi_{n_k}\}$ converges uniformly into $v(\xi)$ on any bounded neighborhood in R^2 . Thus, $v(\xi) \in S_0$, $v_1'(\xi) \geq 0$ and $v_2'(\xi) \leq 0$. For any $\xi \in R$, there in fact exist $v_1'(\xi) > 0$ and $v_2'(\xi) < 0$. Similarly, it is clear that $\lim_{\xi \rightarrow +\infty} v(\xi) = E_1$ and $\lim_{\xi \rightarrow +\infty} v'(\xi) = (0, 0)$. From the uniform convergence of $\{\phi_{n_k}\}$ to $v(\xi)$,

it has $\lim_{\xi \rightarrow -\infty} v(\xi) = (\bar{H}, \bar{Q})$, where $\frac{u_*}{2} \leq \bar{H} \leq u_*$, $\frac{w_*}{2} \leq \bar{Q} \leq w_*$. Applying the above theorem, it has $\lim_{\xi \rightarrow -\infty} v'(\xi) = (0, 0)$, $\lim_{\xi \rightarrow -\infty} v''(\xi) = (0, 0)$. From Eq. 9, it follows that

$p_1(\bar{H}, \bar{Q}) = 0$, $p_2(\bar{H}, \bar{Q}) = 0$. Thus, $(\bar{H}, \bar{Q}) = E_*$, that is $\lim_{\xi \rightarrow -\infty} v(\xi) = E_*$. The above results show that solution $v(\xi)$ goes to the interior equilibrium E_* as $t \rightarrow -\infty$. This completes the proof.

Numerical Simulations

In this part, numerical simulations are performed to establish the existence of traveling wave solutions of Eq. 2. All the chosen parameters are $\alpha = 0.7$, $s = 0.5$, $a = 9.5$, $b = 3$ and $\delta = 8$. Then Eq. 2 possesses the equilibrium points: $E_0 = (0, 0)$, $E_1 = (1, 0)$, $E_2 = (0, 3)$, $E_3 = (0.7394, 3.7394)$. By Theorem 1, there exists a traveling wave solution, which connects E_1 with E_3 . Using the numerical simulation, the traveling wave solutions of Eq. 2 are shown in Fig. 1 and Fig. 2. It should be stressed that the trajectory of traveling wave solutions moves from E_1 to E_3 . This confirms the feasibility and correctness of the theoretical analyses.

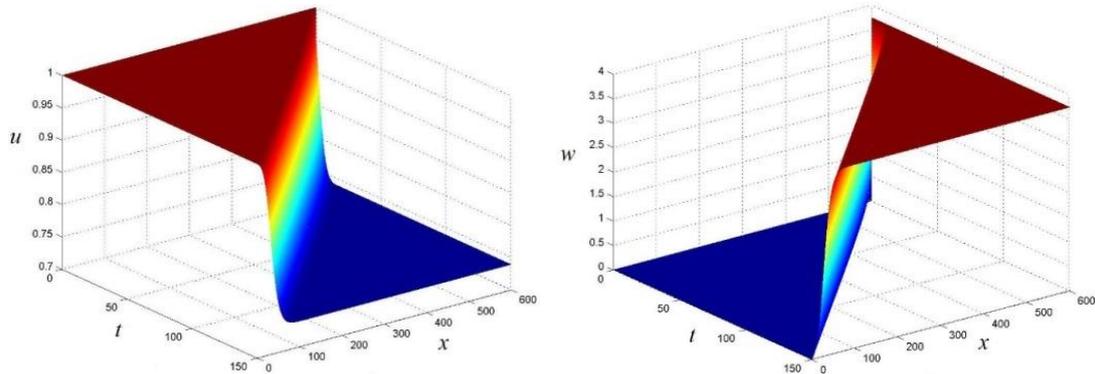


Figure 1. Traveling wave solutions found by numerical integration of Eq.2.

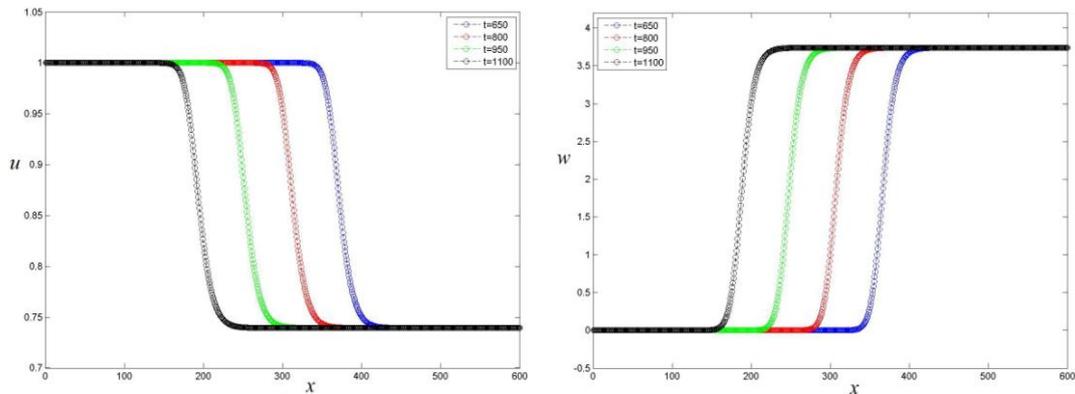


Figure 2. Profiles of traveling wave solutions.

Conclusions

In this paper, our studied focus of the modified Leslie-Gower model with spatial diffusion and Holling IV schemes is its traveling wave solutions. First, we prove the existence of traveling wave connecting E_1 and E_3 in Eq. 2. On the basis of the above theoretical analyses, the numerical simulations have been to show the correctness of analytic results.

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