The Reconstruction of Spring-Mass System with Partial Given Data

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Keywords: Spring-mass system; jacobi matrix; inverse eigenvalue problem; eigenpair

Abstract: In this paper, two inverse vibration problems of constructing a grounding spring-mass system from its two eigenpairs and part of spring stiffness are considered. The vibration system is constrained to satisfy a relation that the total mass of system is a constant, and the problems are transferred into inverse eigenvalue problems for Jacobi matrix. The necessary and sufficient conditions for the construction of physically realizable systems with positive parameters are derived. Furthermore, the corresponding numerical algorithms and numerical example are given.

Introduction

Spring-mass systems are the basic dynamic systems. Many inverse vibration problems can be divided into inverse vibration problems of spring-mass systems by using lumped mass method or finite difference method. The vibration includes the longitudinal vibration of rod, the lateral vibration of string, the torsional vibration of hub disk and so on. Inverse vibration problems for spring-mass systems, generally speaking, are how to determine the physical elements of the systems from part natural frequencies (eigenvalues) or vibration modes (eigenvectors) or some physical parameters. Related research has important application in vibration control, structural design, parameters identification, etc. The problems are transferred into inverse eigenvalue problems for Jacobi matrices in mathematics. Recently, some new results have been obtained on the inverse eigenvalue problems for Jacobi matrices, see [1-4]. Using two sets of eigenvalues or two incomplete eigenpairs, the inverse vibration problems of spring-mass systems have been studied by Nylen and Uhlig [5], and Huang, et al. [6]. Bai [7], and Tian and Dai[8] considered by one eigenpair or two eigenpairs to determine spring-mass systems, and proposed numerical algorithms for solving the problems. In view of practical engineering problems, this paper studies two classes of inverse vibration problems that are generalization of the problem in [8], and constructs the grounding spring-mass system from its two eigenpairs, some physical parameters and the total mass of system. The necessary and sufficient conditions of unique solution for the two problems are proved, moreover, the expressions of the solution and the related numerical algorithms are derived.

Assume that anterior p particles of a spring-mass system are connected to the ground by springs (Fig. 1). Generalized eigenvalue equation for the system is: $KX = \lambda MX$, where $\lambda = \omega^2$, ω is natural frequency, X is vibration mode, λ and X are respectively eigenvalue and eigenvector of matrix pair (K,M), particle quality $m_i > 0$, ungrounded spring stiffness $k_i > 0 (i = 1,2,\cdots,n-1)$, grounded spring stiffness $c_j > 0 (j = 1,2,\cdots, p)$, mass matrix is: $M = diag(m_1, m_2,\cdots, m_n)$, stiffness matrix is:

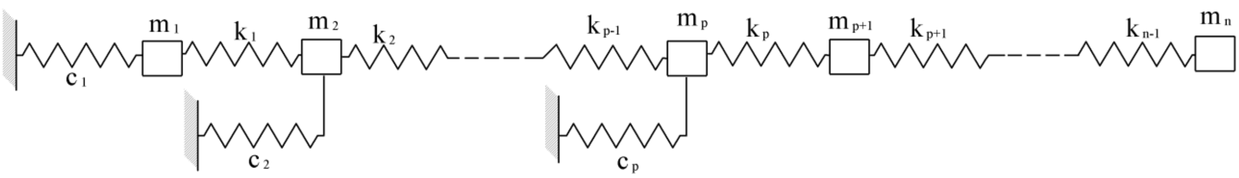

Fig. (1). A grounding spring-mass system

$$K=\begin{bmatrix} c_1+k_1 & -k_1 & & & & & & \\ -k_1 & c_2+k_1+k_2 & -k_2 & & & & & \\ & & \ddots & & & & & \\ & & -k_{p-1} & c_p+k_{p-1}+k_p & -k_p & & & \\ & & & -k_p & k_p+k_{p+1} & -k_{p+1} & & \\ & & & & & \ddots & & \\ & & & & & -k_{n-2} & k_{n-2}+k_{n-1} & -k_{n-1} \\ & & & & & & -k_{n-1} & k_{n-1} \end{bmatrix}.$$

In fact, the generalized eigenvalue equation $KX=\lambda MX$ is equivalent to the standard eigenvalue equation $JX=\lambda X$, where $J=M^{-1}K$ is $n\times n$ Jacobi matrix.

The problems we considered are as follows.

Problem 1 Given two eigenpairs of the system (Fig. 1) (λ,X) ,(μ,Y)($\lambda,\mu\in \boldsymbol{R}^{+}$ and $\lambda\neq\mu$), grounded spring stiffness $\{c_i\}_{i=1}^{p-1}$ and constant C, find the remaining physical parameters of the system, that is, solve particle quality $\{m_i\}_{i=1}^{n}$, ungrounded spring stiffness $\{k_i\}_{i=1}^{n-1}$ and the p$^{\text{th}}$ grounded spring stiffness c_p.

Problem 2 Given two eigenpairs of the system (Fig. 1) (λ,X) ,(μ,Y)($\lambda,\mu\in \boldsymbol{R}^{+}$ and $\lambda\neq\mu$), ungrounded spring stiffness $\{k_i\}_{i=1}^{p-1}$ and constant C, find the remaining physical parameters of the system, that is, solve particle quality $\{m_i\}_{i=1}^{n}$, grounded spring stiffness $\{c_i\}_{i=1}^{p}$ and ungrounded spring stiffness $\{k_i\}_{i=p}^{n-1}$.

In Problem1 and Problem 2, all parameters of the systems should satisfy

$KX=\lambda MX$, $KY=\mu MY$, (1)

and $\sum_{i=1}^{n} m_i = C$, where K and M are respectively stiffness matrix and mass matrix, (λ,X) and (μ,Y) are respectively the i$^{\text{'th}}$ and the j$^{\text{'th}}$ eigenpair of the system $(1\le i^{'}, j^{'}\le n)$, when eigenvalues of the system are arranged in ascending order, and $X=(x_1,x_2,\cdots,x_n)^T$, $Y=(y_1,y_2,\cdots,y_n)^T\in\boldsymbol{R}^n$.

Let $u_i = x_i - x_{i-1}$, $v_i = y_i - y_{i-1}$, $l_i=\lambda x_i v_i - \mu y_i u_i$, $i=1,2,\cdots,n$,

$h_i=\lambda x_i v_{i+1} - \mu y_i u_{i+1}$, $q_i = u_i v_{i+1} - u_{i+1} v_i$, $i=1,2,\cdots,n-1$,

$z_i=(\lambda-\mu)x_i y_i$, $f_i = x_i v_{i+1} - y_i u_{i+1}$, $g_i = x_i v_i - y_i u_i$, $w_i=\dfrac{l_i}{h_{i-1}}w_{i-1}+c_i z_i$, $t_i=\dfrac{q_i}{h_{i-1}}w_{i-1}+c_i f_i$,

$r_i = k_i f_i - k_{i-1} g_i$, $s_i = k_i h_i - k_{i-1} l_i$, $i=1,2,\cdots,p$,

$$d=\frac{f_p}{z_p}+\frac{q_{p+1}}{h_{p+1}}+\sum_{i=p+2}^{n-1}\frac{q_i}{h_i}\Big[\prod_{j=p+1}^{i-1}\frac{l_j}{h_j}\Big]+\frac{u_n}{\lambda x_n}\prod_{j=p+1}^{n-1}\frac{l_j}{h_j},\quad e=C-\sum_{i=1}^{p-1}m_i+\frac{k_{p-1}g_p}{z_p},$$

where $x_0=y_0=0$, $k_0=0$, $w_0=0$.

Solvability of the Problems

In this section, we will discuss the conditions for the solvability of Problem 1 and Problem 2. We first present the following definition and lemma for later discussion.

Definition. (See [9].) For $n\times 1$ real vector $X=(x_1,x_2,\cdots,x_n)^T$, sign change number of sequence $x_1,x_2,\cdots,x_n$ (that value being equal to zero can be neglected) is denoted by $S(X)$.

Lemma. (See [9].) Let $\{\lambda_i\}_{i=1}^{n}$ be eigenvalues of $n\times n$ Jacobi matrix J, with $\lambda_1<\lambda_2<\cdots<\lambda_n$, and let (λ,X) be eigenpair of Jacobi matrix J . Then, (λ,X) is the i$^{\text{th}}$ eigenpair of J if and only if $S(X)=i-1$, $i=1,2,\cdots,n$.

Equations (1) can be rewritten as follows.

$$\begin{cases} \lambda m_i x_i = c_i x_i + k_{i-1} u_i - k_i u_{i+1}, \\ \mu m_i y_i = c_i y_i + k_{i-1} v_i - k_i v_{i+1}, \end{cases} \quad i = 1, 2, \cdots, n, \quad (2)$$

where $k_0 = k_n = 0$, $x_0 = y_0 = 0$, $c_{p+1} = \cdots = c_n = 0$.

Theorem 1. Problem 1 has a unique solution if and only if

(1) $S(X) = i^{'} - 1,\ S(Y) = j^{'} - 1;$

(2) $h_i \neq 0$, and h_i, w_i, t_i have the same sign, $i = 1, 2, \cdots, p-1$; $h_i \neq 0$, and h_i, l_i, q_i have the same sign, $i = p+1, p+2, \cdots, n-1$; $x_n u_n > 0$, $y_n v_n > 0$, $l_n = 0$;

(3) $d \neq 0$, d and e have the same sign;

(4) $z_p \neq 0$, and z_p, r_p, s_p have the same sign.

Proof. For systems of linear equations (2), since $k_0 = 0$, if $i = 1, 2, \cdots, p-1$ then

$\begin{cases} \lambda x_i m_i + u_{i+1} k_i = c_i x_i + u_i k_{i-1}, \\ \mu y_i m_i + v_{i+1} k_i = c_i y_i + v_i k_{i-1}, \end{cases}$ have a unique solution if and only if $\begin{vmatrix} \lambda x_i & u_{i+1} \\ \mu y_i & v_{i+1} \end{vmatrix} \neq 0$,

that is $h_i \neq 0$, $i = 1, 2, \cdots, p-1$, and the solution is $k_i = \dfrac{l_i k_{i-1} + c_i z_i}{h_i}$, $m_i = \dfrac{q_i k_{i-1} + c_i f_i}{h_i}$,

Let $w_i^{'} = l_i k_{i-1} + c_i z_i$, then $k_i = \dfrac{w_i^{'}}{h_i}$, thus, $k_{i-1} = \dfrac{w_{i-1}^{'}}{h_{i-1}}$,

Therefore, $w_i^{'} = \dfrac{l_i}{h_{i-1}} w_{i-1}^{'} + c_i z_i$, Obviously, $w_i^{'}$ and w_i have the same form.

Let $w_i = w_i^{'}$, then $k_i = \dfrac{w_i}{h_i}$. And $q_i k_{i-1} + c_i f_i = \dfrac{q_i}{h_{i-1}} w_{i-1} + c_i f_i = t_i$, thus,

$$k_i = \frac{w_i}{h_i},\ m_i = \frac{t_i}{h_i},\ i = 1, 2, \cdots, p-1. \qquad (3)$$

And for $k_i > 0$, $m_i > 0$, h_i, w_i, t_i have the same sign, $i = 1, 2, \cdots, p-1$.

For systems of linear equations (2), if $i = p$ and assume that k_p is given, then

$\begin{cases} \lambda x_p m_p - x_p c_p = k_{p-1} u_p - k_p u_{p+1}, \\ \mu y_p m_p - y_p c_p = k_{p-1} v_p - k_p v_{p+1}, \end{cases}$ have a unique solution if and only if $\begin{vmatrix} \lambda x_p & -x_p \\ \mu y_p & -y_p \end{vmatrix} \neq 0$,

that is $z_p \neq 0$, and the solution is

$$m_p = \frac{r_p}{z_p},\ c_p = \frac{s_p}{z_p}. \qquad (4)$$

And for $m_p > 0$, $c_p > 0$, z_p, r_p, s_p have the same sign.

In addition, the expressions of m_p and c_p can be expanded as follows.

$$m_p = k_p \frac{f_p}{z_p} - \frac{k_{p-1} g_p}{z_p},\ c_p = k_p \frac{h_p}{z_p} - \frac{k_{p-1} l_p}{z_p}.$$

For systems of linear equations (2), if $i = p+1, p+2, \cdots, n-1$ and assume that k_p is given, then

$\begin{cases} \lambda x_i m_i + u_{i+1} k_i = k_{i-1} u_i, \\ \mu y_i m_i + v_{i+1} k_i = k_{i-1} v_i, \end{cases}$ have a unique solution if and only if $\begin{vmatrix} \lambda x_i & u_{i+1} \\ \mu y_i & v_{i+1} \end{vmatrix} \neq 0$, that is $h_i \neq 0$, and the solution is

$$k_i = k_{i-1} \frac{l_i}{h_i},\ m_i = k_{i-1} \frac{q_i}{h_i}, \qquad i = p+1, p+2, \cdots, n-1. \qquad (5)$$

And for $k_i > 0$, $m_i > 0$, h_i, l_i, q_i have the same sign, $i = p+1, p+2, \cdots, n-1$.

In addition, the expressions of k_i and m_i can be expanded as follows.

$$k_i = k_{i-1}\frac{l_i}{h_i} = k_{i-2}\frac{l_{i-1}}{h_{i-1}}\cdot\frac{l_i}{h_i} = \cdots = k_p\prod_{j=p+1}^{i}\frac{l_i}{h_j},\ i = p+1, p+2,\cdots,n-1,$$

$$m_{p+1} = k_p\frac{q_{p+1}}{h_{p+1}},\quad m_i = k_{i-1}\frac{q_i}{h_i} = k_{i-2}\frac{l_{i-1}}{h_{i-1}}\cdot\frac{q_i}{h_i} = \cdots = k_p\frac{q_i}{h_i}\prod_{j=p+1}^{i-1}\frac{l_i}{h_j}.$$

For systems of linear equations (2), if $i = n$ and assume that k_p is given, then

$\begin{cases}\lambda x_n m_n = k_{n-1}u_n,\\ \mu y_n m_n = k_{n-1}v_p,\end{cases}$ have a unique solution if and only if $m_n = \frac{k_{n-1}u_n}{\lambda x_n} = \frac{k_{n-1}v_n}{\mu y_n}$.

Since $m_n > 0$, $k_{n-1} > 0$, $\lambda > 0$, $\mu > 0$, $x_n u_n > 0$, $y_n v_n > 0$, and $\lambda x_n v_n = \mu y_n u_n$, that is $l_n = 0$.

Therefore, $m_n = \frac{k_{n-1}u_n}{\lambda x_n}$ or $\frac{k_{n-1}v_n}{\mu y_n}$ (6)

In addition, using the recurrence formula of k_{n-1}, the expression of m_n can be expanded as follows.

$m_n = k_{n-1}\frac{u_n}{\lambda x_n} = k_p\frac{u_n}{\lambda x_n}\prod_{j=p+1}^{n-1}\frac{l_i}{h_j}$. From $\sum_{i=1}^{n} m_i = C$ and the above expansions of $m_i\,(i = p, p+1,\cdots,n)$,

we have $\sum_{i=1}^{p-1} m_i + k_p\frac{f_p}{z_p} - \frac{k_{p-1}g_p}{z_p} + k_p\frac{q_{p+1}}{h_{p+1}} + k_p\sum_{i=p+2}^{n-1}\frac{q_i}{h_i}[\prod_{j=p+1}^{i-1}\frac{l_j}{h_j}] + k_p\frac{u_n}{\lambda x_n}\prod_{j=p+1}^{n-1}\frac{l_j}{h_j} = C$,

that is $k_p\{\frac{f_p}{z_p} + \frac{q_{p+1}}{h_{p+1}} + \sum_{i=p+2}^{n-1}\frac{q_i}{h_i}[\prod_{j=p+1}^{i-1}\frac{l_j}{h_j}] + \frac{u_n}{\lambda x_n}\prod_{j=p+1}^{n-1}\frac{l_j}{h_j}\} = C - \sum_{i=1}^{p-1} m_i + \frac{k_{p-1}g_p}{z_p}$,

Therefore, $k_p = \frac{e}{d}$. (7)

Since $k_p > 0$, $d \neq 0$, d and e have the same sign. Combining with above discussion and with lemma, we get the theorem.

It is not difficult to prove Theorem 2 by the same method which we apply to prove Theorem 1.

Theorem 2. Problem 2 has a unique solution if and only if

(1) $S(X) = i' - 1,\ S(Y) = j' - 1$;

(2) $z_i \neq 0$, and z_i, r_i, s_i have the same sign, $i = 1,2,\cdots,p-1$; $h_i \neq 0$, and h_i, l_i, q_i have the same sign, $i = p+1, p+2,\cdots,n-1$; $x_n u_n > 0$, $y_n v_n > 0$, $l_n = 0$;

(3) $d \neq 0$, d and e have the same sign;

(4) $z_p \neq 0$, and z_p, r_p, s_p have the same sign.

When the above-mentioned conditions are satisfied, the remaining physical parameters of spring-mass system have the expressions as follows:

$$m_i = \frac{r_i}{z_i},\ c_i = \frac{s_i}{z_i},\ i = 1,2,\cdots,p-1,\quad m_p = \frac{r_p}{z_p},\ c_p = \frac{s_p}{z_p},\ k_p = \frac{e}{d},\ k_i = k_{i-1}\frac{l_i}{h_i},$$

$$m_i = k_{i-1}\frac{q_i}{h_i},\ i = p+1, p+2,\cdots,n-1,\ m_n = \frac{k_{n-1}u_n}{\lambda x_n}\ \text{or}\ \frac{k_{n-1}v_n}{\mu y_n}.$$

Numerical Method

Based on the above discussion, we write numerical algorithm for solving Problem 1 as follows.

Algorithm 1.

Step 1. Compute $S(X)$, $S(Y)$. If $S(X) \neq i' - 1$ or $S(Y) \neq j' - 1$, go to step 6.

Step 2. Compute $\{u_i\}_{i=1}^{n}$, $\{v_i\}_{i=1}^{n}$, $\{l_i\}_{i=1}^{n}$, $\{h_i\}_{i=1}^{n-1}$, $\{q_i\}_{i=1}^{n-1}$, $\{f_i\}_{i=1}^{p}$, $\{z_i\}_{i=1}^{p}$, $\{w_i\}_{i=1}^{p-1}$, $\{t_i\}_{i=1}^{p-1}$.

Step 3. If some $h_i = 0$, $i = 1,2,\cdots, p-1, p+1,\cdots,n-1$, go to step 6;
If h_i, w_i, t_i ($i = 1,2,\cdots, p-1$) have different sign, go to step 6;
If h_i, l_i, q_i ($i = p+1, p+2,\cdots,n-1$) have different sign, go to step 6;
If $x_n u_n \le 0$ or $y_n v_n \le 0$ or $l_n \ne 0$, go to step 6.
Step 4. Compute $\{k_i\}_{i=1}^{p-1}$, $\{m_i\}_{i=1}^{p-1}$, d, e. If $d = 0$ or d, e have different sign, go to step 6.
Step 5. Compute k_p, r_p, s_p. If $z_p = 0$ or z_p, r_p, s_p have different sign, go to step 6.
Step 6. The solution can not be determined uniquely, end the algorithm.
Step 7. Compute m_p, c_p, $\{k_i\}_{i=p+1}^{n-1}$, $\{m_i\}_{i=p+1}^{n-1}$, m_n.

Example 1. Given $\lambda = 1.0508$, $\mu = 2.1186$, $n = 8$, $p = 3$, $\{c_i\}_{i=1}^{2} = \{3,2\}$,
$X = (-0.6041,-0.5581,-0.4094,-0.3459, -0.0691,0.1601,0.0170,-0.0754)^T$,
$Y = (-0.1911,0.1296,0.4811,0.2395, -0.3766,0.3127,-0.5350,0.3635)^T$, and $\sum_{i=1}^{8} m_i = C = 35$,
construct K and M such that (λ, X) and (μ, Y) are respectively the 3rd and the 6th eigenpair of the system.

By Algorithm 1, we get $S(X) = 2$, $S(Y) = 5$.
$\{u_i\}_{i=1}^{8} = \{-0.6041,0.0460,0.1487,0.0635, 0.2768,0.2292,-0.1431,-0.0924\}$,
$\{v_i\}_{i=1}^{8} = \{-0.1911,0.3207,0.3515,-0.2416, -0.6161,0.6893,-0.8477,0.8985\}$,
$\{l_i\}_{i=1}^{8} = \{-0.1233,-0.2007,-0.3028,0.0556, 0.2656,-0.0359,-0.1773,0\}$,
$\{h_i\}_{i=1}^{7} = \{-0.1850,-0.2470,0.0392,0.0835, 0.1328,-0.0478,-0.0887\}$,
$\{q_i\}_{i=1}^{7} = \{-0.1849,-0.0315,-0.0582,0.0278, 0.3320,-0.0957,-0.2069\}$,
$\{f_i\}_{i=1}^{3} = \{-0.1849,-0.2154,0.0684\}$, $\{z_i\}_{i=1}^{3} = \{-0.1233,0.0772,0.2103\}$,
$\{w_i\}_{i=1}^{2} = \{-0.3698,-0.2468\}$, $\{t_i\}_{i=1}^{2} = \{-0.5548,-0.4939\}$.
By (3), we have $\{k_i\}_{i=1}^{2} = \{1.9995,0.9995\}$, $\{m_i\}_{i=1}^{2} = \{2.9999,1.9999\}$,
hence $d = 9.6480$, $e = 28.9764$. By (7), we obtain $k_3 = 3.0033$, hence $r_3 = 0.4206$, $s_3 = 0.4204$.
By (4), (5), (6), we have $m_3 = 2.0001$, $c_3 = 1.9989$, $\{k_i\}_{i=4}^{7} = \{2, 3.9991, 3.0012, 6.0016\}$,
$\{m_i\}_{i=4}^{7} = \{0.9984, 4.9993, 8.0012, 7.0021\}$, $m_8 = 6.9992$.

Using Matlab, it is easy to get that all generalized eigenvalues of $KX = \lambda MX$ are
$\sigma(K,M) = \{0.0247,0.3799,1.0508,1.3319, 1.9117,2.1188,3.0928,6.4857\}$.

The eigenvector which corresponds to eigenvalue $\lambda = 1.0508$ is
$X = (-0.6041,-0.5581,-0.4094,-0.3459, -0.0691,0.1601,0.0170,-0.0754)^T$,
and the eigenvector which corresponds to eigenvalue $\mu = 2.1188$ is
$Y = (-0.1909,0.1295,0.4807,0.2392, -0.3764,0.3127,-0.5354,0.3639)^T$.

Obviously, $\sum_{i=1}^{8} m_i = 35.0001$. The numerical value explain that the algorithm 1 is efficient.

The algorithm is presented for solving Problem 2 as follows.

Algorithm 2.
Step 1. Compute $S(X)$, $S(Y)$. If $S(X) \ne i' - 1$ or $S(Y) \ne j' - 1$, go to step 6.
Step 2. Compute $\{u_i\}_{i=1}^{n}$, $\{v_i\}_{i=1}^{n}$, $\{l_i\}_{i=1}^{n}$, $\{h_i\}_{i=1}^{n-1}$, $\{z_i\}_{i=1}^{p}$, $\{f_i\}_{i=1}^{p}$, $\{g_i\}_{i=1}^{p}$, $\{r_i\}_{i=1}^{p-1}$, $\{s_i\}_{i=1}^{p-1}$, $\{q_i\}_{i=p+1}^{n-1}$.
Step 3. If some $z_i = 0$ or z_i, r_i, s_i ($i = 1,2,\cdots, p-1$) have different sign, go to step 6;
If some $h_i = 0$ or h_i, l_i, q_i ($i = p+1, p+2,\cdots,n-1$) have different sign, go to step 6;
If $x_n u_n \le 0$ or $y_n v_n \le 0$ or $l_n \ne 0$, go to step 6.

Step 4. Compute $\{m_i\}_{i=1}^{p-1}$, $\{c_i\}_{i=1}^{p-1}$ d , e. If $d=0$ or d , e have different sign, go to step 6.
Step 5. Compute k_p, r_p, s_p. If $z_p=0$ or z_p, r_p, s_p have different sign, go to step 6.
Step 6. The solution can not be determined uniquely, end the algorithm.
Step 7. Compute m_p, c_p, $\{k_i\}_{i=p+1}^{n-1}$, $\{m_i\}_{i=p+1}^{n-1}$, m_n.

Conclusion

This paper discusses the constructional problems for the spring-mass system whose anterior p particles are connected to the ground. Two eigenpairs and anterior p-1 grounded spring stiffness or anterior p-1 ungrounded spring stiffness to determine the real vibration system are solved under the total mass of system constraint. The necessary and sufficient conditions for the existence and uniqueness of the solution are obtained. The results of numerical example show that the proposed algorithm work well.

Acknowledgment

This work is supported by the Key Scientific Research Project of Hubei Provincial Department of Education (No. D20134301), the Teaching Research Project of Jingchu University of Technology (No. JX2014-19).

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