

Probing The Labelling and Isomorphic Graph of Generalized Fibonacci Caterpillar Tree

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Abstract. “All trees are graceful” has not been proven. Applying the construction method to show every generalized Fibonacci caterpillar tree is a set-ordered odd graceful graph meanwhile every generalized Fibonacci caterpillar tree not only have an odd-elegant labelling but also possess a super total graceful labelling. On this basis, we prove that the isomorphic graph of generalized Fibonacci caterpillar tree is a set-ordered graceful graph as well as a super total graceful graph. Finally, we show each generalized Fibonacci caterpillar sun-tree is a felicitous graph.

1 Introduction

As everyone know, graph labelling theory plays a important role in coding theory, complex networks, logistics and other fields. In operations research or systems theory and practice of systems engineering. By applying graph labellings, one can divide large systems into subsystems. One, very often, utilize graph labellings to distinguish vertices and edges between vertices for the sake of searching certain fast algorithms to simulate efficient transmissions and communications in information spreading networks. There is a large and growing literature on graph labeling [1]. Rosa [2] testified if G is graceful as well as overall its vertices are even defrees, and $|E(G) \equiv 0|$ or $3 \pmod{4}$. Cycle C_n are graceful if and only if $n \equiv 0$ or $3 \pmod{4}$; paths P_n ; wheels W_n and complete bipartite graphs $K_{m,n}$ are graceful whereas complete graphs K_n are graceful if and only if $n \leq 4$. It has been conjectured that all trees are graceful, it is still an open problem. Rosa contributed various labeling for solving problems of graph decementation and proposed some entertaining open questions. In 1979, Bermond [3] made conjecture: all lobster trees are graceful graphs, this conjecture has not been proven or negated. Sin-Min Lee [4] and others in 1991 proposed a conjecture “Every tree is a felicitous tree”. This conjecture has the equal theoretical worth to “Every tree is a graceful tree” and also has the same difficulty to be proved. Both of them are NP-problems that have drew many researchers to be proved [5-9]. Many scholars attacked mathematical conjectures and made enornoms contribution on graph theory. These works prompt the graph labelling theory fleetly become a dynamic branch of graph labellings. In other words, graph labelling theory has played a important role in many fields.

In this paper the method of constructing the two graceful labelling proved that a new class of generalized Fibonacci caterpillar tree and its isomorphic graph is a set-ordered graceful graph, on this basis and prove that they are also super total graceful graph, then prove the tree T stand for the generalized Fibonacci caterpillar isomorphic graph, if T is consist of sun-trees we say that T is a felicitous graph and it is short of GFCST graph.

We employ criterion symbol and technology of graph theory. Graphs make reference to loopless, no multiple edges, undirected, linked and limited, in addition to elsewhere specified. A (p,q) -graph G is one with p vertices and q edges. The steno symbol $[m, n]$ indicates an integer set $\{m, m+1, \dots, n\}$, where m and n both are integers with $0 \leq m < n$; the mark $[s, t]^o$ figures an odd-set $\{s, s+1, \dots, t\}$, where s and t both are odd integers with $1 \leq s < t$; and the mark $[k, l]^e$ figures an even-set $\{k, k+2, \dots, l\}$, where integers k and l both are even with $0 \leq k < l$.

Definition 1.^[10,11] The graph G has p vertices and q edges, then we grants a mapping $f: V(G) \rightarrow [0, 2q-1]$, and for distinct $v, u \in V(G)$ which satisfied $f(u) \neq f(v)$; About for every edges vu of G , there

correspond label $f(vu)$ satisfied $f(vu)=|f(v)-f(u)|$, and we can get that $f(vu)=[1, 2q-1]^0$ for every $vu \in E(G)$. So we name f is an *odd-graceful labeling* as well as the graph G is known as *odd-graceful*.

Definition 2.^[12] The G has p vertices and q edges and is a bipartite graph. (V_1, V_2) is the two bipartition of G . Suppose G exists an odd-graceful labeling f which satisfied $\max\{f(y) \mid y \in Y\} < \min\{f(x) \mid x \in X\}$, then we say f is a *set-ordered odd-graceful labeling*.

Definition 3.^[13,14] The graph G has p vertices and q edges, then we grants a mapping $f: V(G) \rightarrow [0, 2q-1]$, and for distinct $v, u \in V(G)$ which satisfied $f(u) \neq f(v)$; but for every $uv \in E(G)$, there correspond label $f(vu)$ satisfied $f(vu)=f(v)+f(u) \pmod{2q}$ as well as $f(vu)=[1, 2q-1]^0$, so we say f is an *odd-elegant labeling*, G is known as *odd-elegant graph*.

Definition 4.^[15,16] A total graceful labeling (TGL) of a (p, q) graph G is a bijection f from $V(G) \cup E(G)$ to the set $\{1, 2, 3, \dots, p+q\}$ so that $f(uv)=|f(u)-f(v)|$ for all $uv \in E(G)$. A TGL is called a super TGL if $f(E)=\{1, 2, 3, \dots, q\}$. A graph that admits a super total graceful labeling is called a *super total graceful graph* (STGG).

Definition 5.^[6] The graph G has p vertices and q edges, then we grants a mapping $f: V(G) \rightarrow [0, q]$, and for distinct $v, u \in V(G)$ which satisfied $f(v) \neq f(u)$. A felicitous labelling called f of graph G satisfied that for distinct $v, u \in V(G)$, $f(v) \neq f(u)$, and for every edge $vu \in E(G)$, there have $f(vu)=f(v)+f(u) \pmod{q}$ and $f(u)=[0, q-1]$. Then we call G is a *felicitous graph*.

Definition 6. The G has p vertices and q edges and is a bipartite graph. (V_1, V_2) is the two bipartition of G . Suppose G exists a felicitous labelling f which satisfied $\max\{f(y) \mid y \in Y\} < \min\{f(x) \mid x \in X\}$, now we named f is a *set-ordered felicitous labelling*, G is known as a *set-ordered felicitous graph*, and write this case as $f(X) < f(Y)$.

For a path $p=a_0a_1a_2, \dots, a_n$, we add new vertices $a_{i,1}, a_{i,2}, \dots, a_{i,m_i}$ to each vertex a_i by edges a_ia_{ij} with $i=1, 2, \dots, n$ and $j=1, 2, \dots, m_i$ ($m_i \geq 0$), call such tree a caterpillar tree. For a caterpillar tree T , if $m_1=a, m_2=b$ ($a \geq b \geq 1$) and $m_i=m_{i-2}+m_{i-1}$ for $i \geq 2$ hold true, we call T a generalized Fibonacci caterpillar tree, and denoted by F_n , a_0 is called the s -head of F_n as well as a_{n,m_n} the L -head of F_n .

Let T_i is a copy of q generalized Fibonacci caterpillar tree F_n , $i=1, 2, \dots, \beta$. Join the L -head a_{n,m_n}^i of the tree T_i with the s -head a_0^{i+1} of the tree T_{i+1} by an edge ($i=1, 2, \dots, \beta-1$), and joining the L -head of the tree T_β with the s -head of the tree T_1 with an edge. And the resultant tree is named a generalized Fibonacci caterpillar sun-tree (GFCST).

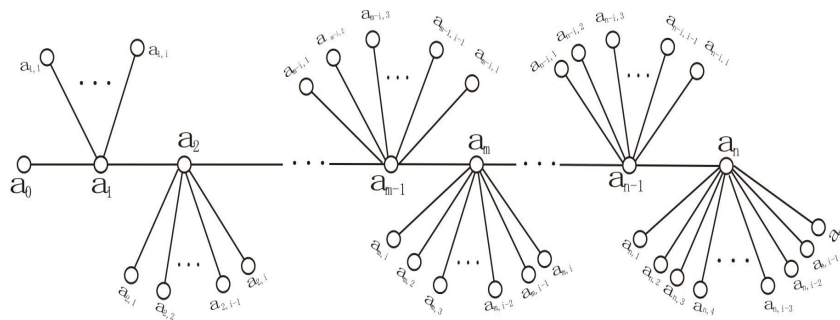


Fig 1. Generalized Fibonacci caterpillar tree.

2 Main Results

Theorem 1. Every generalized Fibonacci caterpillar tree F_n is a set-ordered odd graceful tree.

Proof. Let F_n be a generalized Fibonacci caterpillar tree defined Section 1. We label the vertices of each generalized Fibonacci's caterpillar graphs via a labelling f established in the following:

$$\begin{aligned} f(a_n) &= 0; f(a_{n,n}) = 2q-1, f(a_{n,j}) = 2q-1-2(m_n-j), j \in \{1, 2, \dots, m_n\}; \\ f(a_{n-1}) &= f(a_{n,1})-2, f(a_{n-1,j}) = f(a_n) + 2(m_{n-1}-j+1), j \in \{1, 2, \dots, m_{n-1}\}; \\ f(a_{n-2}) &= f(a_{n-1,1})+2, f(a_{n-2,j}) = f(a_{n-1}) - 2(m_{n-2}-j+1), j \in \{1, 2, \dots, m_{n-2}\}; \\ f(a_{n-3}) &= f(a_{n-2,1})-2, f(a_{n-3,j}) = f(a_{n-2}) + 2(m_{n-3}-j+1), j \in \{1, 2, \dots, m_{n-3}\}; \end{aligned}$$

$$\begin{aligned} f(a_{n-4}) &= f(a_{n-3,1}) + 2, f(a_{n-4,j}) = f(a_{n-3}) - 2(m_{n-4} - j + 1), j \in \{1, 2, \dots, m_{n-4}\}; \\ f(a_{n-5}) &= f(a_{n-4,1}) - 2, f(a_{n-5,j}) = f(a_{n-4}) + 2(m_{n-5} - j + 1), j \in \{1, 2, \dots, m_{n-5}\}; \\ f(a_{n-6}) &= f(a_{n-5,1}) + 2, f(a_{n-6,j}) = f(a_{n-5}) - 2(m_{n-6} - j + 1), j \in \{1, 2, \dots, m_{n-6}\}; \end{aligned}$$

In general, on the basis of the above, we can easily find the following equation:

$$\begin{aligned} f(a_k) &= f(a_{k+1,1}) - 2, f(a_{k,j}) = f(a_{k+1}) + 2(m_k - j + 1), j \in \{1, 2, \dots, m_k\}; \\ f(a_{k-1}) &= f(a_{k,1}) + 2, f(a_{k-1,j}) = f(a_k) - 2(m_{k-1} - j + 1), j \in \{1, 2, \dots, m_{k-1}\}. \end{aligned}$$

And then we also find that equation: $f(a_k) = f(a_{k+1,1}) - 2, f(a_{k,j}) = f(a_{k+1}) + 2(m_k - j + 1), j \in \{1, 2, \dots, m_k\}$

to show vertices which labeling are odd numbers, so we can note the set make up by this vertices is Y , as well as the equation: $f(a_{k-1}) = f(a_{k,1}) + 2, f(a_{k-1,j}) = f(a_k) - 2(m_{k-1} - j + 1), j \in \{1, 2, \dots, m_{k-1}\}$

to show vertices which labeling are even numbers, so we can note the set make up by this vertices is X .

So that, we can see $f(a_n) = 0, f(a_{0,1}) = 0$ and $f(a_{n,n}) = 2q - 1$. Clearly, $f(x_i)$ is even $i \in [1, s]$; and $f(y_j)$ is odd $j \in [1, t - 1]$, and $\max\{f(x) \mid x \in X\} < \min\{f(y) \mid y \in Y\}$. $f(x_{i+1}) < f(x_i)$ for $i \in [1, s]$; $f(y_{j+1}) < f(y_j)$ for $j \in [1, t - 1]$, and The certification is completed.

Theorem 2. If generalized Fibonacci caterpillar tree F_n is set-ordered odd-graceful graph, then it is odd-elegant graph.

Proof. According to Theorem 1, we know every generalized Fibonacci caterpillar tree F_n exists a set-ordered odd-graceful labelling f . Now we regulate one new labelling h of F_n as below:

$$\begin{aligned} h(a_n) &= f(a_n) = 0; h(a_{n,n}) = 2q - f(a_{n,n}) = 1, h(a_{n,j}) = 2(m_n - j) + 1, j \in \{1, 2, \dots, m_n - 1\}; \\ h(a_{n-1}) &= 2(q + 1) - f(a_{n,1}), h(a_{n-1,j}) = f(a_{n-1,j}), j \in \{1, 2, \dots, m_{n-1}\}; \\ h(a_{n-2}) &= f(a_{n-2}), h(a_{n-2,j}) = 2q + 2(m_{n-2} - j + 1) - f(a_{n-1}), j \in \{1, 2, \dots, m_{n-2}\}; \\ h(a_{n-3}) &= 2(q + 1) - f(a_{n-2,1}), h(a_{n-3,j}) = f(a_{n-3,j}), j \in \{1, 2, \dots, m_{n-3}\}; \\ h(a_{n-4}) &= f(a_{n-4}), h(a_{n-4,j}) = 2q + 2(m_{n-4} - j + 1) - f(a_{n-3}), j \in \{1, 2, \dots, m_{n-4}\}; \\ h(a_{n-5}) &= 2(q + 1) - f(a_{n-4,1}), h(a_{n-5,j}) = f(a_{n-5,j}), j \in \{1, 2, \dots, m_{n-5}\}; \\ h(a_{n-6}) &= f(a_{n-6}), h(a_{n-6,j}) = 2q + 2(m_{n-6} - j + 1) - f(a_{n-5}), j \in \{1, 2, \dots, m_{n-6}\}. \end{aligned}$$

Above all, we can promote formula to general case:

$$\begin{aligned} h(a_k) &= f(a_k), h(a_{k,j}) = 2q + 2(m_k - j + 1) - f(a_{n+1}), j \in \{1, 2, \dots, m_k\}; \\ h(a_{k+1}) &= 2(q + 1) - f(a_{k+2,1}), h(a_{k+1,j}) = f(a_{k+1,j}), j \in \{1, 2, \dots, m_{k+1}\}. \end{aligned}$$

Furthermore,

$$\begin{aligned} h(a_k) &= f(a_k), h(a_{k+1,j}) = f(a_{k+1,j}), j \in \{1, 2, \dots, m_{k+1}\} \text{ is even. } h(a_{k,j}) = 2q + 2(m_k - j + 1) - f(a_{n+1}), j \in \{1, 2, \dots, m_k\}, \\ h(a_{k+1}) &= 2(q + 1) - f(a_{k+2,1}) \text{ is odd. And } h(E(F_n)) = 2q - f(E(F_n)), f(E(F_n)) = [1, 2q - 1]^o, \text{ then we have:} \\ h(E(F_n)) &= [1, 2q - 1]^o. \text{ In consequence, } h \text{ is an odd-elegant labelling of } F_n. \text{ (See figure 2)} \end{aligned}$$

Theorem 3. Every generalized Fibonacci caterpillar tree F_n is the super total graceful tree.

Proof. According to Theorem 1, let f be a set-ordered graceful labeling of the graph. Then we define a new labeling g , such that:

$$\begin{aligned} g(a_n) &= p; g(a_{n,n}) = 2q - 1 + p, g(a_{n,j}) = 2q - 1 - 2(m_n - j) + p, j \in \{1, 2, \dots, m_n\}; \\ g(a_{n-1}) &= g(a_{n,1}) - 2 + p, g(a_{n-1,j}) = g(a_n) + 2(m_{n-1} - j + 1) + p, j \in \{1, 2, \dots, m_{n-1}\}; \\ g(a_{n-2}) &= g(a_{n-1,1}) + 2 + p, g(a_{n-2,j}) = g(a_{n-1}) - 2(m_{n-2} - j + 1) + p, j \in \{1, 2, \dots, m_{n-2}\}; \\ g(a_{n-3}) &= g(a_{n-2,1}) - 2 + p, g(a_{n-3,j}) = g(a_{n-2}) + 2(m_{n-3} - j + 1) + p, j \in \{1, 2, \dots, m_{n-3}\}; \\ g(a_{n-4}) &= g(a_{n-3,1}) + 2 + p, g(a_{n-4,j}) = g(a_{n-3}) - 2(m_{n-4} - j + 1) + p, j \in \{1, 2, \dots, m_{n-4}\}; \\ &\dots \end{aligned}$$

$$g(a_k) = g(a_{k+1,1}) - 2 + p, g(a_{k,j}) = g(a_{n+1}) + 2(m_k - j + 1) + p, j \in \{1, 2, \dots, m_k\}$$

Hence, g is a super total graceful labeling of F_n . The theorem is covered.

Corollary 1. Every generalized Fibonacci caterpillar isomorphic graph is a set-ordered odd graceful graph also a super total graceful graph.

Theorem 4. Every generalized Fibonacci caterpillar sun-tree (GFCST) is a felicitous graph.

Proof. According to the definition 4, let T_i is a GFCST. Let X_i, Y_j be the bipartition of a bipartite graph G , so that $V(T_i) = X_i \cup Y_j$, where $X_i = \{x_i \in [1, s]\}$ and $Y_j = \{y_j \in [1, t]\}$ with $s + t = |T_i|$. Every T_i acknowledges a set-ordered felicitous labelling f such that: $\max\{f(x_i) \mid x_i \in X_i\} < \min\{f(y_j) \mid y_j \in Y_j\}$. Without loss of generality, we have $f(x_k) = k - 1, k \in [1, s]$ and $f(y_j) = j + s - 1, j \in [1, t]$ when $i \in [1, n]$, join in an edge between vertex x_i^1 and y_{i+1}^1 , and connect x_s^n and x_1^1 with an edge, then we can obtain a GFCST in Theorem 3. Defined a new labelling g for GFCST as follow:

$$g(x_k^i) = \frac{1}{2}(s+t)(i-1) + k - 1, i \in [1, n]^o, k \in [1, s]; g(y_j^i) = \frac{1}{2}(s+t)(n+i) + \frac{1}{2}t(i-2) + j - 1, i \in [1, n]^l, j \in [1, t];$$

$$g(x_k^i) = \frac{1}{2}s(n+1) + \frac{1}{2}i(s+t) + \frac{1}{2}2t(n-1) - k, \quad i \in [1, n]^o, k \in [1, s]; \quad g(y_j^i) = \frac{1}{2}i(s+t) - r, \quad i \in [1, n]^l, j \in [1, t];$$

$$g(x_k^i y_j^i) = g(x_k^i) + g(y_j^i), \quad i \in [1, n]^o, k \in [1, s], j \in [1, t];$$

now, it is easily testify the set of vertex labelling satisfies

$$\{g(x_k^i) \mid i \in [1, n]^o, k \in [1, s]\} \cup \{g(y_j^i) \mid i \in [1, n]^l, j \in [1, t]\} = [0, \frac{1}{2}(s+t)(n-1) + s - 1],$$

and

$$\{g(y_j^i) \mid i \in [1, n]^o, j \in [1, t]\} \cup \{g(x_k^i) \mid i \in [1, n]^l, k \in [1, s]\} = [\frac{1}{2}(s+t)(n-1) + s, n(s+t) - 1]$$

we notice that the total vertex is not variable in the process of constructing GFCST. Furthermore the number of edge of GFCST is $|E(G)| = n(s+t)$. So we have

$$g(V(G)) = [0, n(s+t) - 1] \subseteq [0, n(s+t)]$$

and

$$g(E(T_1 \cup T_2 \cup \dots \cup T_n)) = [\frac{1}{2}(s+t)(n-1) + s - 1, \frac{1}{2}(s+t)(3n-1) + s - 2] \subseteq [0, n(s+t) - 1]$$

Judging by the Definition 4, the labelling of g is a felicitous labelling of GFCST. In other words, the GFCST is the felicitous graph. (See figure 3)

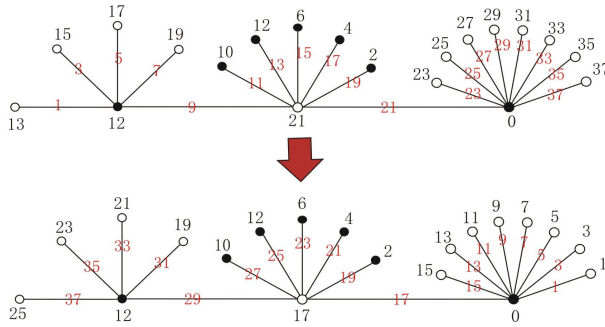


Fig 2. An odd-elegant of generalized Fibonacci caterpillar tree F_3 .

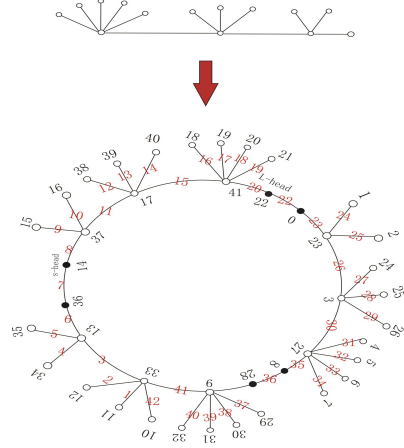


Fig 3. A felicitous labelling of GFCST.

3 Conclusions

Clearly, the results, here, are suitable for constructing graceful trees or bipartite graceful trees. Most of proofs of the results in this note are effective. We have defined a new labelling called generalized Fibonacci caterpillar tree labelling, in order to study more complex models from observation and analysis of graphs. First, we validate the generalized Fibonacci caterpillar tree not only is set-ordered odd graceful tree but also is an odd-elegant graph meanwhile is super total graceful tree. Besides, we give a proof for illustrating set-ordered odd graceful labellings as well as odd-elegant labelling and super total graceful labelling. Finally, we provide the method for stating the isomorphic graph of generalized Fibonacci caterpillar tree labelling sun-tree is a felicitous graph. It could be available to indicate more complicated graph labelling that acknowledge graceful labellings and GFCST-labellings, and to discover some underlying applications of these labellings. As a byproduct it is easy to distinguish new classes of graceful graphs by using graceful similar systems built upon known smaller graceful trees, which are gracefully similar. It could be interesting to probe and identify more related concepts and relationships among them. For example, it could be nice to try to work out the specific conditions when the graph is a strongly graceful.

At the end of this article, we propose several problems for further research as follows

Problem 1. Adding leaves to the set-ordered odd graceful generalized Fibonacci caterpillar trees, could we produce the tree is set-ordered odd graceful and an odd-elegant graph ?

Problem 2. Could we can use the odd-elegant labelling f of generalized Fibonacci caterpillar trees produce a strong set-ordered odd graceful?

Problem 3. Whether all higher-order Fibonacci caterpillar trees and their isomorphic graphs are set-ordered odd graceful, also, super total graceful ?

Problem 4. If every Fibonacci caterpillar trees exist a super total magic labeling, furthermore, or every Fibonacci caterpillar trees admits a super (k, λ) -magically total labelling for some integers k and $\lambda \neq 0$?

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