Coleman automorphisms of the normalizers of finite p-subgroups

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Abstract. Let G be a finite group and P be a p-subgroup of G, where p is arbitrary prime dividing of $|G|$. If P is abnormal in G or the normalizer of P in G is a p-group, then every Coleman automorphism of G is an inner automorphism. Interest in such automorphisms arose from the study of the normalizer problem for integral group rings.

1. Introduction

Let G be a finite group and let σ be an automorphism of G. Recall that σ is said to be a Coleman automorphism if the restriction of σ to any Sylow subgroup of G equals the restriction of some inner automorphisms of G. These automorphisms form a characteristic subgroup $Aut_{Col}(G)$ of automorphisms of G. It is clear that $Inn(G)$ is a normal subgroup of $Aut_{Col}(G)$. Set $Out_{Col}(G) = Aut_{Col}(G)/Inn(G)$, which is said to be Coleman outer automorphism of G. E.C.Dade has studied lot's of Coleman automorphism about finite group in [2], and he proved that $Out_{Col}(G)$ is a nilpotent group. Recently, M. Hertweck and W. Kimmerle proved that $Out_{Col}(G)$ is a commute group in [1], and they got some sufficient conditions of Coleman outer automor-phisms group of G is a p'-group. Moreover, if G is a nilpotent group, then every Coleman automorphism of G is an inner automorphism, i.e., $Out_{Col}(G)=1$. More generally, if G is a quasinilpotent group, M. Hertweck and W. Kimmerle also proved that every Coleman automorphism of G is an inner automorphism.

The aim of the present paper is to investigate the influence of the normalizers of finite p-subgroups on Coleman automorphism. We show some sufficient conditions for $Out_{Col}(G)=1$. We can prove the following main result (Theorem 3.1).

Theorem A. Let G be a finite group and p be arbitrary prime dividing of $|G|$. If every p-subgroup of G is abnormal in G, then every Coleman automorphism of G is an inner automorphism, i.e., $Out_{Col}(G)=1$.

Moreover, we have the following result (Theorem 3.3).

Theorem B. Let G be a finite group and p be arbitrary prime dividing of $|G|$. If every normalizer of p-subgroups of G is a p-group, then every Coleman automorphism of G is an inner automorphism, i.e., $Out_{Col}(G)=1$.

2. Notation and preliminaries

In this section, we first fix some nation and then record some lemmas that will be used in the sequel. Let G be a finite group. We write $conj(x)$ for the inner automorphism of G induced by x via conjugacy, i.e., $y^{conj(x)} = y^x$ for any $y \in G$. Denote by $\pi(G)$ the set of all primes dividing the order of G. Let $H \le G$, $\sigma \in Aut(G)$. We write $\sigma|_H$ for the restriction of σ to H. Let $p \in \pi(G)$. Denote by $P \in Syl_p(G)$ the set of all Sylow p-subgroup of G. Respectively, we denote an unique

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largest normal p-subgroup of G by $O_p(G)$. Other notation used will be mostly standard, refer to [4].

In this paper, we present some results which will be used in the proof of the main theorem.

Definition 2.1. Let G be a group, $H \le G$. H is said to be abnormal in G if, for every $g \in G$, $g \in < H, H^g >$.

Definition 2.2. Let $\sigma \in Aut(G)$, $p \in \pi(G)$. σ is said to be p-center automorphism of G if, there is $P \in Syl_p(G)$, such that $\sigma|_P = id|_P$.

Lemma 2.3. Let G be a simple group. Then every Coleman automorphism of G is an inner automorphism, i.e., $Out_{Col}(G) = 1$.

Proof. If G is an abelian simple group, then G is a quasinilpotent group, there is nothing to prove. Let G be a non-abelian simple group. It is clear that no chief factor of G is isomorphic to C_p for each $p \in \pi(G)$, and $Z(F^*(G)) = 1$. It follows from [1,Theorem 21] that $Out_{Col}(G)$ is a p'-group. By choice of p, then every Coleman automorphism of G is an inner automorphism, i.e., $Out_{Col}(G) = 1$.

Lemma 2.4[1, Theorem 14]. Let G be a simple group. Then there is a prime $p \in \pi(G)$ such that p-central automorphisms of G are inner automorphisms.

Lemma 2.5[4, Theorem 8.10]. let G be a non-trivial finite group. Then G is characteristically simple if and only if G is a direct product of finitely many isomorphic copies of a simple group.

Lemma 2.6[4, Lemma5.13]. Let L be a finite normal subgroup of G, and let $P \in Syl_p(G)$. Then $G = N_G(P)L$.

Lemma 2.7[3, Lemma 2]. Let p be a prime, and σ be a p-power order automorphism of a finite group G. Assume further that there is a normal subgroup K of G, such that σ fixes all elements of K, and that σ induces the identity on the quotient group G/K. Then σ induces the identity on $G/O_p(Z(K))$.

Definition 2.8. A group K is said to be complete if $Z(K) = 1$ and $Aut(K) = Inn(K)$.

Lemma 2.9. S_n is complete for every integer $n \ge 3$ with $n \ne 6$.

For a proof of this result, see [8].

3. Proof of Main Theorems

In this paper, we present proofs for Theorems A and B. For convenience, we record Theorem A here as

Theorem 3.1. Let G be a finite group and p be an arbitrary prime of $\pi(G)$. If every p-subgroup of G is abnormal in G, then every Coleman automorphism of G is an inner automorphism, i.e., $Out_{Col}(G) = 1$.

Proof. Let P be a p-subgroup of G and P be abnormal in G. We firstly shows that $N_G(P) = P$. In fact, it is clear that $P \le N_G(P)$. Next, let $x \in N_G(P)$, then $P^x = P$. Since P is abnormal in G, it follows that $x \in < P, P^x >$, which implies that $x \in P$. Hence $N_G(P) = P$. Let L be a minimal normal subgroup of G. Since p is an arbitrary prime divisor of $|G|$, we may assume that P' is a non-trivial Sylow p-subgroup of L. Note that P' is also abnormal in G, it follows that $N_G(P') = P'$. By Lemma 2.6, we have $G = N_G(P')L$, which implies that $G = L$, that is, G is minimal. Therefore G is a simple group. It follows from Lemma 2.3 that every Coleman automorphism of G is an inner automorphism, i.e., $Out_{Col}(G) = 1$.

Corollary 3.2. Let P be an p-subgroup of a finite group G and p be an arbitrary prime of $\pi(G)$. Whenever $g \in G$, $H \leq G$, and $P \leq H \cap H^g$, which implies that $g \in H$. Then every Coleman automorphism of G is an inner automorphism, i.e., $Out_{Col}(G) = 1$.

Proof. Let $g \in G$, write $H =< P, P^g >$, then $H \leq G$, and P, $P^g \leq H$. It follows that $P \leq H^{g^{-1}}$, which implies that $P \leq H \cap H^{g^{-1}}$. Note that $g^{-1} \in H$, which implies that $g \in < P, P^g >$. Thus P is abnormal in G. It follows from Theorem 3.1 that every Coleman automorphism of G is an inner automorphism, i.e., $Out_{Col}(G) = 1$.

Moreover, we have the following result (Theorem B).

Theorem 3.3. Let P be a p-subgroup of a finite group G and p be an arbitrary prime of $\pi(G)$. If $N_G(P)$ is p-group, then every Coleman automorphism of G is an inner automorphism, i.e., $Out_{Col}(G) = 1$.

Proof. Let σ be an Coleman automorphism of G of p-power order, we have to show that $\sigma \in Inn(G)$. Since G is a finite group, then there are minimal normal subgroups of G.

Suppose that G has a non-trivial soluble minimal normal subgroup M. Then there exists a $q \in \pi(G)$ such that $O_q(M) \neq 1$. Since $O_q(M) \lhd G$, which implies that $N_G(O_q(M)) = G$. By hypothesis, $N_G(O_q(M))$ is a q-group, i.e., G is a q-group. It follows from definition of Coleman automorphism that $\sigma \in Inn(G)$.

Now, let L be a insoluble minimal normal subgroup of G. By Lemma 2.5, we may assume that $L = S_1 \times S_2 \times \cdots \times S_t$ with isomorphic simple group S_i $(i = 1, 2, \cdots t)$. Let $P \in Syl_p(L)$. By Sylow's theorem and note that $N_G(P)$ is a p-group, then there exists a Sylow p-subgroup D of G such that $N_G(P) \leq D$, it follows from definition of Coleman automorphism that there exist some $x \in G$, such that $\sigma|_D = conj(x)|_D$, thus $\sigma conj(x^{-1})|_D = id|_D$, we write $\beta = \sigma conj(x^{-1})$, so $\beta|_D = id|_D$, we obtain that $\beta|_{N_G(P)} = id|_{N_G(P)}$ and $\beta|_P = id|_P$. Since $S_i \cong S_j$, where $i, j = 1, 2, \cdots t$, without loss of generality, we may assume that $S_i^{\beta} = S_i$, thus $\beta \in Aut(S_i)$ $(i = 1, 2, \cdots t)$. By Lemma 2.6, we have $G = N_G(P)L$, which implies that

$$\beta|_{G/L} = id|_{G/L} \tag{1}$$

Let P_i be Sylow p-subgroup of S_i. Without loss of generality, we may assume that $P_i \leq P$, hence $\beta|_{P_i} = id|_{P_i}$ $(i = 1, 2, \cdots t)$. Since S_i is a simple group, it follows from Lemma 2.4 that β is an inner automorphism of S_i, i.e., $\beta|_{S_i} \in Inn(S_i)(i = 1, 2, \cdots t)$. Note that $Inn(S_1 \times S_2 \times \cdots \times S_t) = Inn(S_1) \times Inn(S_2) \times \cdots \times Inn(S_t)$ Therefore $\beta|_{S_1 \times S_2 \times \cdots \times S_t} \in Inn(S_1 \times S_2 \times \cdots \times S_t)$, i.e., $\beta|_L \in Inn(L)$. Then there are some $y \in L$ such that $\beta|_L \in conj(y)|_L$, so $\beta conj(y^{-1})|_L = id|_L$, we write $\theta = \beta conj(y^{-1})$, then

$$\theta|_L = id|_L \tag{2}$$

By (1) and (2),

$$\theta|_{G/L} = id|_{G/L} \tag{3}$$

By (2), (3) and Lemma 2.7, we have

$$\theta|_{G/O_pZ(L)} = id|_{G/O_pZ(L)}. \tag{4}$$

Since L is an insoluble minimal normal subgroup of G, which implies that $Z(L)=1$, so $O_p(Z(L))=1$, by (4), we have $\theta|_G = id|_G$. Hence we obtain that $\theta \in Inn(G)$. Implying further that $\sigma \in Inn(G)$, we are done.

As immediate consequences of Theorem 3.1 and Theorem 3.3, we have the following result:

Corollary 3.4. Let P be an arbitrary Sylow p-subgroup of a finite group G and p be an arbitrary prime of $\pi(G)$. If P is abnormal in G, and $N_G(P_1) \le N_G(P)$, where $P_1 \le P$. Then every Coleman automorphism of G is an inner automorphism, i.e., $Out_{Col}(G)=1$.

Proof. Since P is abnormal in G, it follows from Theorem 3.1 that $N_G(P)=P$. Note that, if $P_1 \le P$, then $N_G(P_1) \le N_G(P)$. By choice of p, normalizer of every p-subgroup of G is p-group. By Theorem 3.3, then every Coleman automorphism of G is an inner automorphism, i.e., $Out_{Col}(G)=1$.

More generally, we have the following result .

Examples 3.5. Let S_n be a symmetric group of degree n, where n is a positive integer. Then every Coleman automorphism of S_n is an inner automorphism, i.e., $Out_{Col}(S_n)=1$.

Proof. Let $p \in \pi(S_n)$ and let σ be an Coleman automorphism of S_n of p-power order. We have to show that σ is an inner automorphism of S_n.

This is clear for $n \le 2$. If $n \ge 3$ with $n \ne 6$, we have S_n is complete group, it follows from Definition 2.8 that, $Aut(S_n)=Inn(S_n)$. So $\sigma \in Inn(S_n)$. It remains to consider the case S_6. Since $|Aut(S_6)/Inn(S_6)|=2$, we may assume that σ is a Coleman automorphism of 2-power order. Let $P \in Syl_2(S_6)$ with $(12) \in P$. Then , by the definition of Coleman automorphism, there exists some $x \in S_6$, such that $\sigma|_P = conj(x)|_P$. In particular, σ sends the transposition (12) of S_6 to some conjugate of it. But it is known that any outer automorphism of S_6 maps the transposition (12) to a permutation of type $(2,2,2)$. Which implies that $\sigma \in Inn(S_6)$, we are done.

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