

A class of new endomorphisms with some properties

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Abstract. In this paper, we give a class of new endomorphisms with some properties based on the concept of morphic endomorphisms, called generalized morphic. We discuss the connection between the generalized morphic and the structure of groups. It is proved that the quasi-cyclic group is a generalized morphic group, rather than a morphic group. At the same time, we give out some applications of generalized morphic.

1. Introduction

G. Ehrlich investigated unit regular endomorphisms, and revealed the closely connection between the morphic endomorphisms and unit regular endomorphisms in [1]. He showed that an endomorphism α of a module ${}_R M$ is unit regular if and only if it is regular and $M/M^\alpha \cong \text{Ker}(\alpha)$. W. K. Nicholson and M. F. Yousif proposed the concept of morphic and it has been extensively studied in module and ring theory respectively in [2] and [3]. They also obtained many famous results. Y. Li, W. K. Nicholson and L. Zan, investigated the concept of morphic in the category of groups, and gave an important connection between morphic endomorphisms and the structure of groups in [4].

In this paper, on the basis of the concept of morphic, we give a new observation of morphic endomorphism, called generalized morphic. Note that a morphic endomorphism is a generalized morphic endomorphism. Conversely, we can prove the following main results:

Theorem A Let G be a quasi-cyclic group. Then G is a generalized morphic group, rather than a morphic group.

As an application of generalized morphic groups, we have:

Theorem B Let G be a generalized morphic group. If $K \triangleleft G$ and $K \cong G$, then $K = G$.

If G is a group, we write $\text{End}(G)$ for the monoid of endomorphisms $\alpha: G \rightarrow G$, and we write $\text{Aut}(G)$ for the group of automorphisms of G . Let α be an endomorphism of G , denote by G^α the image of α , and denote by $\text{Ker}(\alpha)$ the kernel of α . As usual we write N^+ for the set of positive integers. And the other symbols are standard that can refer to [5].

2. Generalized morphic

Definition 2.1. Let G be a group, $\alpha \in \text{End}(G)$. If $G^\alpha \triangleleft G$, then α is said to be a normal endomorphism of G .

Definition 2.2. Let G be a group, $\alpha \in \text{End}(G)$. Then α is called morphic if $G^\alpha \triangleleft G$ and $G/G^\alpha \cong \text{Ker}(\alpha)$; A group G is called morphic if every endomorphism α of G for which G^α is normal in G satisfies $G/G^\alpha \cong \text{Ker}(\alpha)$.

Proposition 2.3. Let G be a group, the following are equivalent for $\alpha \in \text{End}(G)$.

- (1) $G^\alpha \triangleleft G$ and G/G^α can be embedded in $\text{Ker}(\alpha)$.
- (2) There exists $\beta \in \text{End}(G)$ with $G^\beta \leq \text{Ker}(\alpha)$ and $G^\alpha = \text{Ker}(\beta)$.
- (3) There exists $\beta \in \text{End}(G)$ with G^β can be embedded in $\text{Ker}(\alpha)$ and $G^\alpha = \text{Ker}(\beta)$.

Proof. (1) \Rightarrow (2) By (1), then let $\sigma: G/G^\alpha \rightarrow \text{Ker}(\alpha)$ is an injective homomorphism, define

$\beta: G \rightarrow G$ by $g^\beta = (gG^\alpha)^\sigma$. It is clear that $(gG^\alpha)^\sigma \in G$. For arbitrary $g_1, g_2 \in G$, if $g_1 = g_2$, $g_1 g_2^{-1} = 1 \in G$. Therefore $g_1 G^\alpha = g_2 G^\alpha$. Since σ is an injective homomorphism, $(g_1 G^\alpha)^\sigma = (g_2 G^\alpha)^\sigma$. Hence β is well defined. Moreover, since $(g_1 g_2)^\beta = (g_1 g_2 G^\alpha)^\sigma = g_1^\beta g_2^\beta$, $\beta \in \text{End}(G)$. Hence $G^\beta = (G/G^\alpha)^\sigma \leq \text{Ker}(\alpha)$ and $\text{Ker}(\beta) = \{g \in G \mid (gG^\alpha)^\sigma = 1\} = \{g \in G \mid gG^\alpha = G^\alpha\} = G^\alpha$.

(2) \Rightarrow (3) This is obvious.

(3) \Rightarrow (1) Given β as in (3), then $G^\alpha = \text{Ker}(\beta) \triangleleft G$ and $G/G^\alpha = G/\text{Ker}(\beta) \cong G^\beta$. By (3), G/G^α can be embedded in $\text{Ker}(\alpha)$.

Definition 2.4. Let G be a group, $\alpha \in \text{End}(G)$. Then α is called generalized morphic if $G^\alpha \triangleleft G$ and G/G^α can be embedded in $\text{Ker}(\alpha)$; A group G is called generalized morphic if every endomorphism α of G for which G^α is normal in G satisfies G/G^α can be embedded in $\text{Ker}(\alpha)$.

Note that morphic is generalized morphic. Conversely, we can prove the following result.

Definition 2.5. Let C be the field of complex numbers and p be an arbitrary prime where C^\times is the group of units of C . Write $C_{p^\infty} = \{z \in C^\times \mid \text{there is a positive integer } n \text{ such that } z^{p^n} = 1\}$. Then the group C_{p^∞} is called quasi-cyclic groups.

Theorem 2.6. Let $G = C_{p^\infty}$ be a quasi-cyclic group. Then

(1) Every proper subgroup of G is a cyclic group of order p^m . Write $C_{p^m} = \{z \in C^\times \mid z^{p^m} = 1\}$, where m is a non-negative integer.

(2) $G/C_{p^m} \cong G$, $m \geq 2$.

(3) There exists some $\alpha \in \text{End}(G)$ such that $G^\alpha \triangleleft G$ is generalized morphic, but it is not morphic.

Proof. Suppose that J is a proper subgroup of G , by the well ordering principle for the positive integers, we can let $m = \min\{n \in \mathbb{N}^+ \mid z^{p^n} = 1, \text{ for arbitrary } z \in J\}$. Write $C_{p^m} = \{z \in C^\times \mid z^{p^m} = 1\}$. It is clear that $J = C_{p^m}$, i.e., every proper subgroup of G is a cyclic group of order p^m . Write $C_{p^m} = \{z \in C^\times \mid z^{p^m} = 1\}$, where m is a non-negative integer.

Next, we shows that $G/C_{p^m} \cong G$. Let $\varphi: G/C_{p^m} \rightarrow G$ by $(zC_{p^m})^\varphi = z^{p^m}$, $m \geq 2$. Let $zC_{p^m} \in G/C_{p^m}$, $z \in G$. Since G is a group, $z^{p^m} \in G$. For every $z_1^{p^m}, z_2^{p^m} \in G$, if $z_1^{p^m} = z_2^{p^m}$, which implies that $z_1^{-1}z_2 \in C_{p^m}$. Therefore $(z_1^{-1}z_2)^{p^m} = 1$, it follows that $z_1^{p^m} = z_2^{p^m}$, hence φ is well defined. Since $(z_1C_{p^m}z_2C_{p^m})^\varphi = (z_1z_2)^{p^m} = (z_1C_{p^m})^\varphi(z_2C_{p^m})^\varphi$, φ is a homomorphism. Suppose that $z_1^{p^m} = z_2^{p^m}$, then $(z_1^{p^m})^{-1}z_2^{p^m} = 1$. It obvious that $z_1^{-1}z_2 \in C_{p^m}$, which implies that $z_1C_{p^m} = z_2C_{p^m}$, thus φ is injective. For an arbitrary $z \in G$, there exists positive integer i , such that $z^{p^i} = 1$. There also exists $x \in C^\times$ such that $x^{p^m} = z$, which implies that $(x^{p^m})^{p^i} = 1$. Hence $x \in G$, it follows that $x^\varphi = x^{p^m} = z$, thus φ is surjective. Moreover, φ is isomorphic, that is, $G/C_{p^m} \cong G$.

Let $\gamma: G \rightarrow G/C_{p^m}$ be the natural homomorphism. Write $\alpha = \gamma\varphi$, given φ as in (2). Then $\alpha: G \rightarrow G$ is an surjective homomorphism and $G^\alpha = G \triangleleft G$, thus α is a normal endomorphism and satisfies $\text{Ker}(\alpha) = \{z \in G \mid z^{p^m} = 1\} = C_{p^m} \neq 1$ and $G/G^\alpha \cong G/G = 1$. Hence G/G^α can be embedded in $\text{Ker}(\alpha)$, it follows that α is generalized morphic; Since G/G^α is not isomorphic to $\text{Ker}(\alpha)$, α is not generalized morphic.

Theorem 2.7. Let $G = C_{p^\infty}$ be a quasi-cyclic group. Then G is a generalized morphic group, but it is not a morphic group.

Proof. For any $\alpha \in \text{End}(G)$, it is obvious that $G^\alpha \triangleleft G$. Suppose that G^α is proper normal in G , then there exists non-negative integer m such that $G^\alpha = C_{p^m}$. It follows from the fundamental theorem on homomorphisms that $\text{Ker}(\alpha) = G$. Hence α is the trivial endomorphism. It is clear that $G/G^\alpha \cong \text{Ker}(\alpha)$, which implies that α is morphic and is also generalized morphic. Assume that $G^\alpha = 1$, it follows from the fundamental theorem on homomorphisms that $\text{Ker}(\alpha) = G$. Hence α is the trivial endomorphism which implies that α is morphic and is also generalized morphic. Suppose that $G^\alpha = G$, by the fundamental theorem on homomorphisms and theorem 1.6 (2), $\text{Ker}(\alpha) = C_{p^m} \neq 1$ or $\text{Ker}(\alpha) = 1$. If $\text{Ker}(\alpha) = 1$, then $G/G^\alpha \cong \text{Ker}(\alpha)$, hence α is morphic and is also generalized morphic. If $\text{Ker}(\alpha) = C_{p^m} \neq 1$, then G/G^α can be embedded in $\text{Ker}(\alpha)$. Therefore α is generalized morphic. However G/G^α is not isomorphic to $\text{Ker}(\alpha)$, hence α is not generalized morphic. Therefore we conclude that G is a generalized morphic group, rather than a morphic group.

Normal endomorphisms is not necessarily generalized morphic.

Example 2.8. Let G denote the group $C_2 \times C_4$. Then there exists $\alpha \in \text{End}(G)$ such that $G^\alpha \triangleleft G$, but α is not generalized morphic.

Proof. Write $C_2 = \langle a \rangle$ and $C_4 = \langle b \rangle$. Define $\alpha: C_2 \rightarrow C_4$ by $(a^k)^\alpha = b^{2k}$, where $k \in \mathbb{N}^+$. It is clear that α is a homomorphism. Define $\eta: G \rightarrow G$ by $(x, y)^\eta = (1, x^\alpha)$, then $\eta \in \text{End}(G)$ and $G^\eta \cong 1 \times C_2 \triangleleft G$, it follows that η is a normal endomorphism. By [5], we have $G/G^\eta \cong (C_2 \times C_4)/(1 \times C_2) \cong C_2 \times C_2$, $\text{Ker}(\eta) = \{(x, y) \in G \mid (x, y)^\eta = 1\} = \{(x, y) \in G \mid x^\alpha = 1\} = 1 \times C_4$. Therefore G/G^η cannot be embedded in $\text{Ker}(\eta)$, which implies that α is not generalized morphic.

3. The application of generalized morphic

Proposition 3.1. Let $\alpha \in \text{End}(G)$ be generalized morphic. If α is one-to-one, then it is onto.

Proof. Suppose that $\alpha \in \text{End}(G)$ is generalized morphic, then G/G^α can be embedded in $\text{Ker}(\alpha)$. Since α is one-to-one, G/G^α can be embedded in $\text{Ker}(\alpha) = 1$, it follows that $G/G^\alpha = 1$. Thus $G = G^\alpha$, which implies that α is onto. In particular, $\alpha \in \text{Aut}(G)$.

Proposition 3.2.^[4] If $\alpha \in \text{End}(G)$ is generalized morphic, so are $\alpha\rho$ and $\rho\alpha$ for every automorphism ρ of G .

Corollary 3.3. Let α, β are generalized morphic. If α is one-to-one, β is onto, then $\alpha\beta, \beta\alpha$ are generalized morphic.

Example 3.4. Let G be a finite group, $K \triangleleft G$. If $K \cong G$, then $K = G$. Note that the result would fail if G be an infinite group.

Proof. Write $G = \mathbb{Z}^+$ and $K = 2\mathbb{Z}^+$, then $2\mathbb{Z}^+ \triangleleft \mathbb{Z}^+$. Let $\varphi: 2\mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ by $(2n)^\varphi = n$. It clear that φ is isomorphic, which implies that $2\mathbb{Z}^+ \cong \mathbb{Z}^+$ but $2\mathbb{Z}^+ \neq \mathbb{Z}^+$.

Theorem 3.5. Let G be generalized morphic, If $K \triangleleft G$ and $K \cong G$, then $K = G$.

Proof. Since $K \cong G$, let $\rho: G \rightarrow K$ be an isomorphism, define $\alpha: G \rightarrow G$ by $g^\alpha = g^\rho$. Certainly $\alpha \in \text{End}(G)$. We have $\text{Ker}(\alpha) = \{g \in G \mid g^\rho = 1\} = 1$ and $G^\alpha = G^\rho = K \triangleleft G$. Since G is generalized morphic, $G/G^\alpha = G/K$ can be embedded in $\text{Ker}(\alpha)$, it follows that $G/K = 1$, that is, $G = K$.

Theorem 3.6. Let G_1, G_2 be two isomorphic groups. If G_1 is generalized morphic, then G_2 is also generalized morphic.

Proof. Let $\varphi: G_1 \rightarrow G_2$ be an isomorphism and let $\beta \in \text{End}(G)$ such that $G_2^\beta \triangleleft G_2$. It is obvious that $\varphi\beta\varphi^{-1} \in \text{End}(G_1)$. Since G_1 is a generalized morphic group, $G_1^{\varphi\beta\varphi^{-1}} = (G_2^\beta)^{\varphi^{-1}} \triangleleft G_1$ and $G_1 / G_1^{\varphi\beta\varphi^{-1}}$ can be embedded in $\text{Ker}(\varphi\beta\varphi^{-1})$. Let $\psi: G_1 \rightarrow G_2 / G_2^\beta$ by $x^\psi = x^\varphi G_2^\beta$. For all $x_1, x_2 \in G_1$, $(x_1 x_2)^\psi = (x_1 x_2)^\varphi G_2^\beta = x_1^\varphi x_2^\varphi G_2^\beta = (x_1)^\psi (x_2)^\psi$, hence ψ is a homomorphism. For any $x G_2^\beta \in G_2 / G_2^\beta$, $x \in G_2$, because φ be an isomorphism, there exists $y \in G_1$ such that $y^\varphi = x$. Hence $y^\psi = x G_2^\beta$, which implies that ψ is a surjective homomorphism. Now

$$\text{Ker}(\psi) = \{x \in G_1 \mid x^\psi = 1\} = \{x \in G_1 \mid x^\varphi G_2^\beta = G_2^\beta\} = \{x \in G_1 \mid x \in (G_2^\beta)^{\varphi^{-1}}\} = G_1^{\varphi\beta\varphi^{-1}} \quad (1)$$

then $G_1 / \text{Ker}(\psi) \cong G_1 / G_1^{\varphi\beta\varphi^{-1}} \cong G_2 / G_2^\beta$.

Define $\theta: \text{Ker}(\varphi\beta\varphi^{-1}) \rightarrow \text{Ker}(\beta)$ by $y^\theta = y^\varphi$. For all $y \in \text{Ker}(\varphi\beta\varphi^{-1})$, $y^{\varphi\beta\varphi^{-1}} = 1$. Hence $y^{\varphi\beta} \in \text{Ker}(\varphi^{-1}) = 1$, which implies that $y^\varphi \in \text{Ker}(\beta)$, that is, θ is well defined. For arbitrary $y_1, y_2 \in \text{Ker}(\varphi\beta\varphi^{-1})$, $(y_1 y_2)^\theta = (y_1 y_2)^\varphi = y_1^\varphi y_2^\varphi = y_1^\theta y_2^\theta$. Hence θ is a homomorphism. Because $\text{Ker}(\theta) = 1$, so that θ is injective. For any $y \in \text{Ker}(\beta) \leq G_2$, there exists $x \in G$ such that $x^\varphi = y$. Since $x^{\varphi\beta\varphi^{-1}} = ((x^\varphi)^\beta)^{\varphi^{-1}} = (y^\beta)^{\varphi^{-1}} = 1$, $x \in \text{Ker}(\varphi\beta\varphi^{-1})$. It follows that θ is surjective, therefore θ is isomorphic, that is, $\text{Ker}(\varphi\beta\varphi^{-1}) \cong \text{Ker}(\beta)$. Hence G_2 / G_2^β can be embedded in $\text{Ker}(\varphi\beta\varphi^{-1})$, so that G_2 is generalized morphic.

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