Some properties on the homomorphism of groups

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Abstract. In this paper, based on the basic of morphic groups and quasi-morphic groups, we defined a new kind of homomorphism with some properties, called dual homomorphism. We give a necessary and sufficient condition of the dual homomorphism. Moreover, we get a connection between the dual homomorphism and the structure of groups.

1.Introduction

W.K.Nicholson and M.F.Yousuf [1] have defined the concept of the morphic endomorphism, and applied this concept to the structure of ring. In 2010, Y.Li, W.K.Nicholson and L.Zan [2] applied the morphic endomorphism to the group structure, and gave some important connections between them. Q.Wang and K.Long [3] have generalized the concept of the morphic endomorphism on quasi-morphic, and showed that all finite abelian groups are quasi-morphic groups. In this paper, based on the basic of morphic groups and quasi-morphic groups, we continue to study a new group homomorphism with some properties, which is called dual homomorphism. Furthermore, we get that dual homomorphism is a generalization of morphic endomorphism, while it is different from quasi-morphic. And we studied some properties of dual homomorphism.

2. Notations and Preliminaries

In this paper, if G is a group, we write $End(G)$ for the set of endomorphism of G and write $Aut(G)$ for the group of automorphisms of G. Let G_1 and G_2 be groups, we write $Hom(G_1,G_2)$ for the set of homomorphism of G_1 to G_2, and $Iso(G_1,G_2)$ for the isomorphism. If $\alpha \in End(G)$, denote the image of group G under the α by G^{α}, and denote the kernel of α by $Ker(\alpha)$. Moreover, other notations are mostly standard, please refer to [4].

Definition 2.1 Let G_1 and G_2 be groups, $\varphi \in Hom(G_1,G_2)$ is called normal homomorphism if $G_1^{\varphi} \lhd G_2$. In particular, we call φ is normal endomorphism of G_1 if $G_1 = G_2$.

Definition 2.2 let G be a group, $\alpha \in End(G)$ is called morphic endomorphism if it satisfies the condition of $G^{\alpha} \lhd G$ and $G/G^{\alpha} \cong Ker(\alpha)$. Moreover, G is called morphic group if every normal endomorphism α of G is morphic endomorphism.

Definition 2.3 let G be a group, the normal endomorphism α of G is called quasi-morphic if there exists normal endomorphism β and γ such that $Ker(\alpha) = G^{\beta}$ and $G^{\alpha} = Ker(\gamma)$.

Definition 2.4 Let G_1 and G_2 be groups, $\alpha \in Hom(G_1,G_2)$ is called dual homomorphism if there exists $\beta \in Hom(G_2,G_1)$ such that $Ker(\alpha) = G_2^{\beta}$ and $Ker(\beta) = G_1^{\alpha}$. Moreover, we say that G_1 and G_2 are dual if every $\alpha \in Hom(G_1,G_2)$ is dual homomorphism.

As a direct corollary of Definition 2.4, we have

Corollary 2.5 Assume G_1 and G_2 are dual, if $G_1 = G_2$, then G_1 is a morphic group.

Proof Suppose $\alpha \in Hom(G_1, G_2)$, since G_1 with G_2 are dual, then α is a dual homomorphism. According to the definition of dual homomorphism that there exists $\beta \in Hom(G_2, G_1)$ such that $Ker(\alpha) = G_2^{\beta}$ and $Ker(\beta) = G_1^{\alpha}$,thus $G_1^{\alpha} \lhd G_2$ and $G_2 / G_1^{\alpha} \cong Ker(\alpha)$. By $G_1 = G_2$, which implies that $G_1^{\alpha} \lhd G_1$ and $G_1 / G_1^{\alpha} \cong Ker(\alpha)$. Hence α is a morphic of G_1. So G_1 is a morphic group.

Proposition 2.6 Let α be a quasi-morphic of G, which means that there exist normal endomorphism β and γ such that $Ker(\alpha) = G^{\beta}$ and $G^{\alpha} = Ker(\gamma)$. If $\beta = \gamma$, then α is a dual homomorphism.

Lemma 2.7[4] Let $K \lhd G$ and $\alpha \in Aut(G)$. Write $K^{\alpha} = \{k^{\alpha} \mid k \in K\}$. Then $K \cong K^{\alpha} \lhd G$ and $G / K \cong G / K^{\alpha}$.

Lemma 2.8[4] Let $\varphi : G \to H$ be a homomorphism and let $K = Ker(\varphi) \lhd G$. Let γ be the natural homomorphism of G onto G / K. Then there is an injective homomorphism $\psi : G / K \to H$ such that $\varphi = \gamma\psi$. In particular, $\mathrm{Im}(\varphi) \cong G / Ker(\varphi)$.

3.Main Results

Theorem 3.1 Let G_1 and G_2 be groups and let $\varphi \in Hom(G_1, G_2)$. Then φ is dual homomorphism if and only if $G_1^{\varphi} \lhd G_2$ and $G_2 / G_1^{\varphi} \cong Ker(\varphi)$.

Proof Suppose φ is dual homomorphism, then these exists $\beta \in Hom(G_2, G_1)$ such that $Ker(\varphi) = G_2^{\beta}$ and $Ker(\beta) = G_1^{\varphi}$. Moreover, $Ker(\beta) \lhd G_2$, we get $G_1^{\varphi} \lhd G_2$. By Lemma 2.8, we have $G_2 / G_1^{\varphi} = G_2 / Ker(\beta) \cong G_2^{\beta} \cong Ker(\varphi)$, thus $G_2 / G_1^{\varphi} \cong Ker(\varphi)$.

Conversely, suppose that $G_1^{\varphi} \lhd G_2$ and $G_2 / G_1^{\varphi} \cong Ker(\varphi)$, then there exists an isomorphism $f : G_2 / G_1^{\varphi} \to Ker(\varphi)$. Let $\beta : G_2 \to G_1$ ($g^{\beta} = (\overline{g})^f$, for every $g \in G_2$, $\overline{g} = gG_1^{\varphi}$). It is easy to prove that β is well defined. Since f is an isomorphism, it is obvious that $(gh)^{\beta} = (\overline{gh})^f = (ghG_1^{\varphi})^f = (gG_1^{\varphi})^f (hG_1^{\varphi})^f = g^{\beta} h^{\beta}$ for arbitrary $g, h \in G_2$, then $\beta \in Hom(G_2, G_1)$. Furthermore, we have $G_2^{\beta} = (G_2 / G_1^{\varphi})^f = Ker(\varphi)$ and

$$Ker(\beta) = \{g \in G_2 \mid g^{\beta} = 1\} = \{g \in G_2 \mid (gG_1^{\varphi})^f = 1\} = \{g \in G_2 \mid gG_1^{\varphi} = G_1^{\varphi}\} = G_1^{\varphi} \qquad (1)$$

Hence, φ is dual homomorphism by Definition 2.4.

Corollary 3.2 Let G_1 and G_2 be groups, $\varphi \in Hom(G_1, G_2)$. Then the following two statements are equivalent.

(1) φ is dual homomorphism.

(2) These exists $\beta \in Hom(G_2, G_1)$ such that $Ker(\varphi) \cong G_2^{\beta}$ and $Ker(\beta) = G_1^{\varphi}$.

Proof Suppose φ is dual homomorphism, then these exists $\beta \in Hom(G_2, G_1)$ such that $Ker(\varphi) = G_2^{\beta}$ and $Ker(\beta) = G_1^{\varphi}$. It is clear that $Ker(\varphi) \cong G_2^{\beta}$ and $Ker(\beta) = G_1^{\varphi}$.

Conversely, if there exists $\beta \in Hom(G_2, G_1)$ such that $Ker(\varphi) \cong G_2^{\beta}$ and $Ker(\beta) = G_1^{\varphi}$. It is obvious that $G_1^{\varphi} \lhd G_2$. Moreover, it is enough to show that $G_2 / G_1^{\varphi} = G_2 / Ker(\beta) \cong G_2^{\beta} \cong Ker(\varphi)$ by Lemma 2.8. Then φ is dual homomorphism following Theorem 3.1.

Corollary 3.3 Let G_1 and G_2 be groups and let $\alpha \in Iso(G_1, G_2)$. Then α is dual homomorphism.

Proof Since $\alpha \in Iso(G_1, G_2)$, $Ker(\alpha) = 1$ and $G_1^{\alpha} = G_2$. Thus $G_1^{\varphi} \lhd G_2$ and $G_2 / G_1^{\varphi} \cong Ker(\varphi)$. According to Theorem 3.1, α is dual homomorphism.

Theorem 3.4 The composition of dual homomorphism is not necessary dual homomorphism.

Proof Let $G = C_2 \times C_4$, write $C_2 = \langle x \rangle$ and $C_4 = \langle y \rangle$. Suppose that $\pi : G \to G$

($(x,y)^{\pi}=(x,1)$) and $\gamma: G\to G$ ($(x,y)^{\gamma}=(y^{\theta},x^{\sigma})$) are distinct homomorphisms of G, where $\theta: C_4\to C_2$ by $(y^k)^{\theta}=x^k$ and $\sigma: C_2\to C_4$ by $(x^k)^{\sigma}=y^{2k}$ are homomorphisms. It is obvious that $G^{\pi}\lhd G$ and $G/G^{\pi}\cong C_4\cong Ker(\pi)$, then π is dual homomorphism. Similarly, it is easy to get $G^{\gamma}\lhd G$ and $G/G^{\gamma}\cong C_2\cong Ker(\gamma)$, thus γ is dual homomorphism. Note that $\pi\gamma: G\to G$ ($(x,y)^{\pi\gamma}=(1^{\theta},x^{\sigma})=(1,x^{\sigma})$) is well defined by the definition of π and γ. Therefore $G^{\pi\gamma}=1\times C_2\lhd G$ and $Ker(\pi\gamma)=\{(x,y)\in G\mid (x,y)^{\pi\gamma}=1\}=\{(x,y)\in G\mid (1,x^{\sigma})=1\}=1\times C_4$.

Since $G/G^{\pi\gamma}\cong C_2\times C_2$ and $Ker(\pi\gamma)=1\times C_4$, $G/G^{\pi\gamma}$ is not isomorphic to $Ker(\pi\gamma)$. Hence, $\pi\gamma$ is not dual homomorphism.

Corollary 3.5 Let G_1, G_2, G_3 be groups. Suppose that $\varphi\in Hom(G_1,G_2)$ is a dual homomorphism and $\psi\in Hom(G_2,G_3)$ is an isomorphism. Deduce that $\varphi\psi$ is a dual homomorphism of G_1 to G_3. In particular, if $\psi\in Aut(G_2)$, then $\varphi\psi$ is a dual homomorphism of G_1 to G_2.

Proof Since $\varphi\in Hom(G_1,G_2)$ is a dual homomorphism, it is clear that $G_1^{\varphi}\lhd G_2$ and $G_2/G_1^{\varphi}\cong Ker(\varphi)$. Because ψ is an isomorphism , it is enough to show that $\varphi\psi\in Hom(G_1,G_3)$ and $(G_1^{\varphi})^{\psi}=G_1^{\varphi\psi}\lhd G_2^{\psi}=G_3$. Following Lemma 2.7 and ψ is an isomorphism, then $G_3/G_1^{\varphi\psi}\cong G_3/G_1^{\varphi}\cong G_2/G_1^{\varphi}\cong Ker(\varphi)$. Now we just prove that $Ker(\varphi)=Ker(\varphi\psi)$. Actually, for arbitrary $x\in Ker(\varphi)$, $x^{\varphi}=1$. So $x^{\varphi\psi}=1^{\psi}=1$, hence $x\in Ker(\varphi\psi)$, thus $Ker(\varphi)\subseteq Ker(\varphi\psi)$. Similarly, we have $y^{\varphi\psi}=1$ for arbitrary $y\in Ker(\varphi\psi)$, then $y^{\varphi\psi}=y^{\varphi}=1$, $y\in Ker(\varphi)$, thus $Ker(\varphi)\supseteq Ker(\varphi\psi)$. We have $G_3/G_1^{\varphi\psi}\cong Ker(\varphi)=Ker(\varphi\psi)$. By Theorem 3.1, φψ is a dual homomorphism of G_1 to G_3.

In particular, if $\psi\in Aut(G_2)$, let $G_3=G_2$. Similarly, we get that $\varphi\psi$ is a dual homomorphism of G_1 to G_2.

Theorem 3.6 Let G_1, G_2, H_1, H_2 be groups. Moreover, suppose that $G_1\cong G_2$ and $H_1\cong H_2$. If G_1 and H_1 are dual, then G_2 and H_2 are dual.

Proof Let $\varphi: G_1\to G_2$ and $\psi: H_1\to H_2$ be distinct isomorphisms. Suppose that β is a normal homomorphism of G_2 to H_2. We will prove that $H_2/G_2^{\beta}\cong Ker(\beta)$. Obviously, $\varphi\beta\psi^{-1}\in Hom(G_1,H_1)$. Since G_1 and H_1 are dual, it follows that $G_1^{\varphi\beta\psi^{-1}}\lhd H_1$ and $H_1/G_1^{\varphi\beta\psi^{-1}}$ $\cong Ker(\varphi\beta\psi^{-1})$.

Let $\theta: H_1\to H_2/G_2^{\beta}$ ($x^{\theta}=x^{\psi}G_2^{\beta}$, $x\in H_1$). Then θ is well defined, and it is easy to check θ is a surjective homomorphism. By Lemma 2.8, $H_1/Ker(\theta)\cong H_2/G_2^{\beta}$ and

$$Ker(\theta)=\{x\in H_1\mid x^{\theta}=1\}=\{x\in H_1\mid x^{\psi}G_2^{\beta}=G_2^{\beta}\}=\{x\in H_1\mid x\in G_2^{\beta\psi^{-1}}\}=G_1^{\varphi\beta\psi^{-1}} \qquad (2)$$

Thus $H_2/G_2^{\beta}\cong H_1/Ker(\theta)=H_1/G_1^{\varphi\beta\psi^{-1}}\cong Ker(\varphi\beta\psi^{-1})$.

Let $\Phi: Ker(\varphi\beta\psi^{-1})\to Ker(\beta)$, where $x^{\Phi}=x^{\varphi}$. For any $x\in Ker(\varphi\beta\psi^{-1})$, $x^{\varphi\beta\psi^{-1}}=1$. Since ψ is isomorphism, $x^{\varphi\beta}=1$. We obtain $x^{\varphi}\in Ker(\beta)$, i.e. Φ is well defined. For arbitrary $x_1,x_2\in Ker(\varphi\beta\psi^{-1})$, since φ is isomorphism, we have $(x_1x_2)^{\Phi}=(x_1x_2)^{\varphi}=x_1^{\varphi}x_2^{\varphi}=x_1^{\Phi}x_2^{\Phi}$. Hence, Φ is homomorphism. For any $y\in Ker(\beta)\le G_2$, there exists $x\in G_1$ such that $x^{\varphi}=y$. Since $x^{\varphi\beta\psi^{-1}}=((x^{\varphi})^{\beta})^{\psi^{-1}}=(y^{\beta})^{\psi^{-1}}=1^{\psi^{-1}}=1$, it follows that $x\in Ker(\varphi\beta\psi^{-1})$. So Φ is surjective. Furthermore, $Ker(\Phi)=\{x\in Ker(\varphi\beta\psi^{-1})\mid x^{\Phi}=x^{\varphi}\}=1$. So we have that Φ is injective. Therefore

Φ is an isomorphism. Thus $H_2 / G_2^{\beta} \cong Ker(\beta)$, i.e. G_2 and H_2 are dual.

The following corollary identifies a direct result of theorem 3.6.

Corollary 3.7 Let G_1 and G_2 be two isomorphic groups. If G_1 is morphic group so is G_2.

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