# Some Properties of *f*-distributor

Zhang Peixin[1,a,*] Hai Jinke[2,b] and Lv Ruizhen[3,c]

[1,2,3]College of Mathematics, Qingdao University, Shandong 266071, P.R.China
[a]mrzhangpeixin@163.com, [b]haijinke@qdu.edu.cn, [c]1360679962@qq.com

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**Abstract.** In this paper, based on the study of *f*-distributor in [1], we defined the concept of *f*-center and discussed some properties of *f*-center. As an application of these properties, we generalize some properties of *p*-center and center of groups.

## 1 Introduction

I. Hawthorn and Y. Guo put forward the concept of *f*-distributor in [2], and some basic properties about *f*-distributor are discussed. Let $G$ be a group, the center of $G$ is defined to be

$$Z(G)=\{g\in G\,|\,[g,x]=1, \text{ for all } x\in G\}.$$

The *p*-center of $G$ is defined to be

$$Z_p(G)=\{g\in G\,|\,[g,x]_p=[x,g]_p=1, \text{ for all } x\in G\}.$$

Let $f:G\to G$ is a function, and $f(1)=1$. Then the *f*-center of $G$ is defined as

$$Z_f(G)=\{g\in G\,|\,[g,x]_f=[x,g]_f=1, \text{ for all } x\in G\}.$$

In this paper, we give the relationship between *f*-center of a group, *p*-center of a group and center of a group. First, the *f*-center of $G$ is a subgroup of $G$. In addition, we have promoted some famous results. For example, let $G$ be a group, if $G/Z(G)$ is cyclic then $G$ is abelian group. On the other hand, note that $[g,x]=1$, which implies that $[x,g]=1$. If $[g,x]_f=1$, but $[x,g]_f$ is not always equal 1.

## 2 Notation and Preliminaries

In this section, we first fix some nation and then record some lemmas that will be used in the sequel. Throughout this paper, $p$ always denotes a prime number, $Z$ the ring of integers. $Z^+$ is the additive group of $Z$. $Aut(G)$ denotes the set of all automorphisms of $G$ forms a group with respect to composition of maps. $H$ char $G$ denotes that $H$ is characteristic in $G$. Moreover, other notations are mostly standard, please refer to [1] and [3].

In this paper, we present some results which will be used in the proof of the main theorems.

**Definition 2.1**. Let $G$ be a group, if $(xy)^p=x^py^p$, for all $x,y\in G$ .Then $G$ is called a *p*-abelian group.

**Definition 2.2**. Let $G$ be a group. Then the *p*-commutator of $x$ and $y$ is defined as $[x,y]_p=x^{-p}y^{-p}(xy)^p$, for all $x,y\in G$.

**Definition 2.3**. Let $G$ be a group. Then the commutator of $x$ and $y$ is defined as $[x,y]=x^{-1}y^{-1}xy$, for all $x,y\in G$.

**Definition 2.4**. Let $G$ be a group, and $f:G\to G$ is a function from $G$ to itself. Then the *f*-distributor of $x$ and $y$ is defined as $[x,y]_f=f(y)^{-1}f(x)^{-1}f(xy)$, for all $x,y\in G$.

**Definition 2.5**. Let $G$ be a group, $f:G\to G$ and from $G$ to itself. We say that the *f*-distributor commuting if $[x,y]_f=[y,x]_f$, for all $x,y\in G$.

**Definition 2.6**. Let $G$ be a group, and $f: G \to G$ is a function from $G$ to itself, $f(1)=1$. Then the *f*-center of a group is defined as

$$Z_f(G)=\{g \in G \mid [g,x]_f=[x,g]_f=1, \text{ for all } x \in G\}.$$

In particular, if $f: G \to G(x \to x^p)$. Then the *f*-center of $G$ is the *p*-center of $G$,

$$Z_p(G)=\{g \in G \mid [g,x]_p=[x,g]_p=1, \text{ for all } x \in G\}.$$

If $f: G \to G(x \mapsto x^{-1})$. Then the *f*-center of $G$ is the center of $G$,

$$Z(G)=\{g \in G \mid [g,x]_{-1}=[x,g]_{-1}=1, \text{ for all } x \in G\}=\{g \in G \mid gx=xg, \text{ for all } x \in G\}.$$

**Example 2.7**. Let $G=<a,b,c \mid a^{3^2}=b^{3^2}=c^{3^2}=1,[a,b]=c,[b,c]=c^3,[a,c]=1>$, and $f: G \to G$ $(x \to x^3)$ is a function. Then $[b,a]_f=1$, but $[a,b]_f \neq 1$.

**Proof**. Since $[a,b]=c$, $[b,c]=c^3$, $[a,c]=1$, $ab=bac$, $b=cbc^2$, $ac=ca$ .So

$[b,a]_f=1 \Leftrightarrow f(a)^{-1}f(b)^{-1}f(ba)=1 \Leftrightarrow a^{-3}b^{-3}(ba)^3=1 \Leftrightarrow a^{-3}b^{-3}bababa=1 \Leftrightarrow abab=b^2a^2(ba=bac)$
$\Leftrightarrow bacbac=b^2a^2(cb=bc^{-2}) \Leftrightarrow babc^{-2}ac=b^2a^2(ab=bac) \Leftrightarrow bbacc^{-2}ac=b^2a^2 \Leftrightarrow b^2ac^{-1}ac=b^2a^2(ac=ca)$
$\Leftrightarrow b^2a^2=b^2a^2$. Certainly $b^2a^2=b^2a^2$, thus $[b,a]_f=1$.

Moreover $[a,b]_f \neq 1$, in fact, if $[a,b]_f=1$, then

$[a,b]_f=1 \Leftrightarrow f(b)^{-1}f(a)^{-1}f(ab)=1 \Leftrightarrow b^{-3}a^{-3}(ab)^3=1 \Leftrightarrow ababab=a^3b^3 \Leftrightarrow baba=a^2b^2(ba=abc^{-1})$
$\Leftrightarrow abc^{-1}abc^{-1}=a^2b^2(bc^{-1}=cbc) \Leftrightarrow abc^{-1}acbc=a^2b^2(ac=ca) \Leftrightarrow abc^{-1}cabc=a^2b^2 \Leftrightarrow babc=ab^2(ab=bac)$
$\Leftrightarrow babc=bacb \Leftrightarrow bc=cb \Leftrightarrow [b,c]=1 \Leftrightarrow c^3=1$, which is a contradiction to the fact that $c^{3^2}=1$. Hence $[a,b]_f=1$.

**Lemma 2.8**[3]. Every subgroup $H$ of $Z(G)$ is normal in $G$.

## 3 Proof of Mine Theorems

**Theorem 3.1**. Let $G$ be a group. Then $Z_f(G) \leq G$.

**Proof**. First, prove that $f(g^{-1})=f(g)^{-1}$, for all $g \in Z_f(G)$. Let $g \in Z_f(G)$. Then

$$[g^{-1},g]_f=f(g)^{-1}f(g^{-1})^{-1}f(g^{-1}g)=1. \tag{1}$$

Thus $f(g^{-1})f(g)=f(g^{-1}g)=f(1)=1$ . By definition 2.6, this is true for all $g \in Z_f(G)$ , $f(g^{-1})f(g)=1$,so $f(g^{-1})=f(g)^{-1}$. Immediately from the definition1.6 we have $f(1)=1$. Then,

$$[1,x]_f=f(x)^{-1}f(1)^{-1}f(1x)=(f(1)^{-1})^{f(x)}=(1)^{f(x)}=1,$$
$$[x,1]_f=f(1)^{-1}f(x)^{-1}f(x1)=f(1)^{-1}=(1)^{-1}=1. \tag{2}$$

Therefore $[1,x]_f=[x,1]_f=1$, for all $x \in G$. So that $Z_f(G) \neq \varnothing$.

Let $g \in Z_f(G)$. For each $x \in G$, on the one hand $f(gg^{-1}x)=f(x)$. On the other hand $f(gg^{-1}x)=f(g)f(x)$. Then $f(g)f(g^{-1}x)=f(x)$. Left product $f(g)^{-1}$ on both side we have $f(g^{-1}x)=f(g)^{-1}f(x)$. By what we have proved above, $f(g^{-1})=f(g)^{-1}$. Now

$$f(g^{-1}x)=f(g)^{-1}f(x)=f(g^{-1})f(x). \tag{3}$$

Immediate, from the definition of *f*-distributor, $[g^{-1},x]_f=f(x)^{-1}f(g^{-1})^{-1}f(g^{-1}x)=1$. Similarly $[x,g^{-1}]_f=1$. Therefore $g^{-1} \in Z_f(G)$.

Let $g_1,g_2 \in Z_f(G)$. For each $x \in G$, from the definition of $Z_f(G)$, Thus

$$\begin{aligned}[g_1g_2,x]_f &= f(x)^{-1}f(g_1g_2)^{-1}f(g_1g_2x)\\ &= f(x)^{-1}(f(g_1)f(g_2))^{-1}f(g_1)f(g_2)f(x)\end{aligned}$$

$= f(x)^{-1} f(g_2)^{-1} f(g_1)^{-1} f(g_1) f(g_2) f(x)$
$= 1.$ (4)

Similarly $[x, g_1 g_2]_f = 1$. Therefore $g_1 g_2 \in Z_f(G)$. Hence $Z_f(G) \le G$.

**Corollary 3.2**. Let $G$ be a group. For a prime number *p*, $f: G \to G(x \to x^p)$ is a function from $G$ to itself, then $Z_p(G) \le G$.

**Proof**. First, $f(x) = x^p$. By definition 2.6, the *f*-center of $G$ is the *p*-center of $G$, $Z_f(G) = Z_p(G)$. By theorem 3.1, $Z_f(G) \le G$. Hence $Z_p(G) \le G$.

**Corollary 3.3**. Let $G$ be a group, and $f: G \to G(x \mapsto x^{-1})$ is a function from $G$ to itself. Then $Z(G) \le G$.

**Proof**. First, $f(x) = x^{-1}$. By definition 2.6, the *f*-center of $G$ is the center of $G$, $Z_f(G) = Z(G)$. By theorem 3.1, $Z_f(G) \le G$. Hence $Z(G) \le G$.

There are other properties about *f*-center which need to be discussed following.

**Theorem 3.4**. Let $G$ be a group, and $f: G \to G$ is a function from $G$ to itself. If $\alpha f = f \alpha$, for all $\alpha \in Aut(G)$. Then $Z_f(G)$ char G.

**Proof** . By theorem 3.1, $Z_f(G) \le G$. Then $Z_f(G)$ char $G$ if and only if $\alpha(g) \in Z_f(G)$ for all $g \in Z_f(G)$ and $\alpha \in Aut(G)$; that is, if and only if $[\alpha(g), x]_f = [x, \alpha(g)]_f = 1$ for all $x \in G$.

Because $\alpha \in Aut(G)$, then there is $y \in G$ such that $x = \alpha(y)$, and so

$$
\begin{aligned}
[\alpha(g), x]_f &= [\alpha(g), \alpha(y)]_f \\
&= (f(\alpha(y)))^{-1}(f(\alpha(g)))^{-1}(f(\alpha(g)\alpha(y))) = ((f\alpha)(y))^{-1}((f\alpha)(g))^{-1}((f\alpha)(gy)) \\
&= ((\alpha f)(y))^{-1}((\alpha f)(g))^{-1}((\alpha f)(gy)) = (\alpha(f(y)))^{-1}(\alpha(f(g)))^{-1}(\alpha(f(gy))) \\
&= \alpha(f(y)^{-1})\alpha(f(g)^{-1})\alpha(f(gy)) = \alpha(f(y)^{-1} f(g)^{-1} f(gy)) \\
&= \alpha([g, y]_f).
\end{aligned}
$$

Since $g \in Z_f(G)$, $\alpha \in Aut(G)$, $[\alpha(g), x]_f = \alpha([g, y]_f) = \alpha(1) = 1$. Similarly $[x, \alpha(g)]_f = 1$.

Therefore $\alpha(g) \in Z_f(G)$. Hence $Z_f(G)$ char $G$.

As immediate consequences of Theorem 3.4 we have the following result:

**Corollary 3.5**. Let $G$ be a group. For a prime number *p*, $f: G \to G(x \to x^p)$ is a function from $G$ to itself, then $Z_p(G)$ char $G$ .

**Proof**. First, $f(x) = x^p$. For all $x \in G$ and $\alpha \in Aut(G)$,

$$f\alpha(x) = f(\alpha(x)) = \alpha(x)^p = \alpha(x^p) = \alpha(f(x)) = \alpha f(x).$$

So that $\alpha f = f \alpha$. By definition 2.6, the *f*-center of G is the *p*-center of G, $Z_f(G) = Z_p(G)$. Moreover, by theorem 3.4, $Z_f(G)$ char $G$. Hence $Z_p(G)$ char $G$.

**Corollary 3.6**. Let $G$ be a group, and $f: G \to G(x \mapsto x^{-1})$ be a function from $G$ to itself. Then $Z(G)$ char $G$.

**Proof**. First, $f(x) = x^{-1}$. For all $x \in G$, $\alpha \in Aut(G)$,

$$f\alpha(x) = f(\alpha(x)) = \alpha(x)^{-1} = \alpha(x^{-1}) = \alpha(f(x)) = \alpha f(x).$$

So that $\alpha f = f \alpha$. By definition 2.6, the *f*-center of G is the center of G, $Z_f(G) = Z(G)$. Moreover, by theorem 3.4, $Z_f(G)$ char $G$. Hence $Z(G)$ char $G$ .

More generally, we have the following result.

**Theorem 3.7**. Let $G$ be a group, $Z_f(G)$ char $G$, $N \lhd G$, $N \le Z_f(G)$, and $G = \langle x \rangle N$. If $f(x^i) = f(x)^i$, where $i$ is a positive integer . Then *f* is a group homomorphism.

**Proof**. First, $G=<x>N$ . Then $f$ is a group homomorphism if and only if $f(g_1g_2)=f(g_1)f(g_2)$ for all $g_1,g_2\in G$ . By hypothesis $G=<x>N$ . Let $g_1=x^{i_1}z_1$ , $g_2=x^{i_2}z_2$ , and $z_1,z_2\in N$ . Since $N\le Z_f(G)$ , $Z_f(G)$ char $G$ . This is true for all $z_1,z_2,(z_1)^{x^{-i_1}}\in Z_f(G)$ . Thus

$$
\begin{aligned}
f(g_1g_2) &= f(x^{i_1}z_1x^{i_2}z_2)=f(x^{i_1}z_1x^{i_2})f(z_2)=f((z_1)^{-x^{i_1}}x^{i_1}x^{i_2})f(z_2)=f((z_1)^{-x^{i_1}})f(x^{i_1}x^{i_2})f(z_2)\\
&= f((z_1)^{-x^{i_1}})f(x^{i_1+i_2})f(z_2)=f((z_1)^{-x^{i_1}})f(x)^{i_1+i_2}f(z_2)=f((z_1)^{-x^{i_1}})f(x)^{i_1}f(x)^{i_2}f(z_2)\\
&= f((z_1)^{-x^{i_1}})f(x^{i_1})f(x^{i_2})f(z_2)=f((z_1)^{-x^{i_1}}x^{i_1})f(x^{i_2}z_2)=f(x^{i_1}z_1x^{-i_1}x^{i_1})f(x^{i_2}z_2)\\
&= f(x^{i_1}z_1)f(x^{i_2}z_2)=f(g_1)f(g_2).
\end{aligned}
$$

By choice of $g_1$ and $g_2$, then $f$ is a group homomorphism.

Moreover, we have the following result.

**Corollary 3.8**. Let $G$ be a group. For a prime number *p*, $f:G\to G(x\to x^p)$ is a function from $G$ to itself, if $N\lhd G$, $N\le Z_f(G)$ and $G/N$ is cyclic then $G$ is *p*-abelian group.

**Proof**. Firstly, $f(x)=x^p$. By corollary 3.5, $Z_f(G)=Z_p(G)$ char $G$ . Since $G/N$ is cyclic group. Then $G=<x>N$ . Moreover,

$$f(x^i)=(x^i)^p=(x^p)^i=f(x)^i, \text{ for each } i\in Z.$$

Therefore $f(x^i)=f(x)^i$. By theorem 3.7, *f* is a group homomorphism. Thus $f(xy)=f(x)f(y)$. For all x,y$\in G$ , $f(xy)=(xy)^p$ , $f(x)f(y)=(x)^p(y)^p$, and so $(xy)^p=x^py^p$ . By choice of x and y, hence $G$ is *p*-abelian group. Finally, we get the following important corollary.

**Corollary 3.9**. Let $G$ be a group. and $f:G\to G(x\mapsto x^{-1})$ is a function from $G$ to itself, if $N\le Z_f(G)$, and $G/N$ is cyclic then $G$ is abelian group.

**Proof**. First, $f(x)=x^{-1}$. By corollary 3.6, $Z_f(G)=Z(G)$ char $G$ . Since $G/N$ is cyclic then $G=<x>N$ . By hypothesis $N\le Z_f(G)$ and lemma 2.8, so that $N\lhd G$ . Moreover,

$$f(x^i)=(x^i)^{-1}=(x^{-1})^i=f(x)^i, \text{ for each } i\in Z.$$

Therefore $f(x^i)=f(x)^i$. By theorem 3.7, *f* is a group homomorphism. Thus $f(xy)=f(x)f(y)$. For all x,y$\in G$ , $f(xy)=(xy)^{-1}$ , $f(x)f(y)=(x)^{-1}(y)^{-1}=(yx)^{-1}$, and so $(xy)^{-1}=(yx)^{-1}$ , it follow that $xy=yx$. By choice of x and y, hence $G$ is abelian group.

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