A Joint Distribution of Extreme Value for a Renewal Risk Model with Interest Force

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Abstract. In this paper, we consider a renewal risk model with interest force, derive the joint distribution of the minimum surplus and maximum surplus before the ruin, and the integral equation of the joint distribution is also obtained.

Introduction

In recent years, the classic risk model has received a remarkable amount of attention and there have been many generalizations. Sundt and Teugels (1995,1997) considered a compound Poisson model with a constant interest force, and the upper and lower bounds for the ruin probability and the integral equation of the ruin probability were obtained by using renewal techniques. Yang and Zhang (2001a, 2001b, 2001c) used the techniques of Sundt and Teugels (1995), some related problems were obtained. Yang (1998) considered a discrete time risk model with a constant interest force, and both Lundberg-type inequality and non-exponential upper bounds for ruin probabilities were obtained by using martingale inequalities. Renewal risk model with interest force as a generalization of the classic risk model was considered in Wu and Du (2002) by using discrete method. Lin and Wang (2005) adopted a different discrete techniques, derived the distribution of surplus immediately before ruin and that of deficit at ruin, further the integral equations of these distributions were obtained.

In this paper, we consider renewal risk model with interest force. By using the techniques of Lin and Wang (2005), the joint distribution of the minimum surplus and maximum surplus before the ruin is derived, and the integral equation of the joint distribution is also obtained.

Definition of the Model

Let (Ω, F, P) be a complete probability space. We consider the renewal risk model with interest force. Suppose $S(t)$ denote the amount of claim in the time interval $(0,t]$, i.e. $S(t)=\sum_{i=1}^{N(t)} X_i$, where $\{X_i, i\geq 1\}$ is independent and identically distributed (i.i.d.) random variables with common distribution function $F(x)$, denotes the amount of the i th claim. The counting process $\{N(t), t\geq 0\}$ denotes the number of claims up to time t

and is defined as

$N(t)=\max\{k: W_1+W_2+\cdots+W_k \leq t\}$, where the inter-claim times $\{W_i, i\geq 1\}$ are assumed to be i.i.d. random variables with common distribution function $K(w)$. Further, we assume the sequences $\{W_i, i\geq 1\}$ and $\{X_i, i\geq 1\}$ are independent, and that $cE(W_1)>E(X_1)$, providing a positive safety loading factor.

Let $U_\delta(t)$ denotes the insurance company's surplus at time t. From the above assumption, it follows that

$$dU_\delta(t)=cdt+U_\delta(t)\delta dt-dS(t) \tag{1}$$

From Sundt and Teugels (1995) and (1), we know that

$$U_\delta(t) = ue^{\delta t} + c\bar{S}_{\bar{t}|} - \int_0^t e^{\delta(t-v)} dS(v) \tag{2}$$

where

$$\bar{S}_{\bar{t}|} = \int_0^t e^{\delta v} dv = \begin{cases} t & if \quad \delta = 0 \\ \dfrac{e^{\delta t} - 1}{\delta} & if \quad \delta > 0 \end{cases}$$

$u > 0$ is initial surplus of insurance company, $c > 0$ is the premium income of unite time, δ is constant interest force.

Definition1. If $T = \inf\{t > 0 : U_\delta(t) < 0\}$ ($T = \infty$ if the set is empty) , T is the ruin time. Obviously, it's a stopping time.

Definition2. Let $\Psi_\delta(u)$ denote the ultimate ruin probability with initial reserve u , That is

$$\Psi_\delta(u) = P\{\bigcup_{t\ge 0}(U_\delta(t) < 0 \mid U_\delta(0) = u\}$$

Main Results

Let T_n denote the time of the n th claim happening, i.e. $T_n = \sum_{i=1}^{n} W_i$. By (2) we have

$$U_\delta(t) = ue^{\delta t} + c\frac{e^{\delta t} - 1}{\delta} - \sum_{i=1}^{n} X_i e^{\delta(t - T_i)}$$

when $t = T_n$, we have

$$\begin{aligned} U_\delta(T_n) &= ue^{\delta T_n} + c\frac{e^{\delta T_n} - 1}{\delta} - \sum_{i=1}^{n} X_i e^{\delta(T_n - T_i)} \\ &= ue^{\delta T_n} - \sum_{i=1}^{n}[X_i - c\frac{e^{\delta W_i} - 1}{\delta}]e^{\delta(T_n - T_i)} \\ &= ue^{\delta\sum_{i=1}^{n} W_i} - \sum_{i=1}^{n} Y_i e^{\delta(T_n - T_i)} \\ &= ue^{\delta\sum_{i=1}^{n} W_i} - \sum_{i=1}^{n} Y_i e^{\delta\sum_{j=i+1}^{n} W_j} \end{aligned}$$

where

$$Y_i = X_i - c\frac{e^{\delta W_i - 1}}{\delta},\ i \ge 1.$$

Obviously $\{(Y_i, W_i), i \ge 1\}$ are independent and have the same distribution $G(y, w)$,

$$\begin{aligned} G(y,w) &= P\{Y_1 = X_1 - c\frac{e^{\delta W_1} - 1}{\delta} \le y, W_1 \le w\} \\ &= \int_0^w P\{X_1 - c\bar{S}_{\bar{t}|} \le y\} dK(t) \\ &= \int_0^w F(y + c\bar{S}_{\bar{t}|}) dK(t) \end{aligned}$$

For $a > 0, b > 0$, let $H(u, a, b)$

$$H(u,a,b) = P\{\inf_{0\le t<T} U_\delta(t) \ge b, \sup_{0\le t<T} U_\delta(t) \le a, \quad T < \infty \mid U_\delta(0) = u\} \tag{3}$$

be the joint distribution function of the minimum surplus and maximum surplus before the ruin with the initial reserve u .

Theorem 1. Let $H(u, a, b)$ be defined as (3), then we have

（1） when $u > a$ or $u < b$, then $H(u, a, b) = 0$

（2） when $b \le u \le a$， then $H(u,a,b) = \sum_{n=1}^{\infty} h_n(u,a,b)$,

Where

$$h_1(u,a,b) = \int_0^{\infty} \overline{F}(ue^{\delta t} + c\overline{S}_{\bar{t}|})dK(t)$$

$$h_n(u,a,b) = \int_0^{\infty} \int_{ue^{\delta t}+c\overline{S}_{\bar{t}|}-a}^{ue^{\delta t}+c\overline{S}_{\bar{t}|}-b} h_{n-1}(ue^{\delta t} + c\overline{S}_{\bar{t}|} - y, a, b)dF(y)dK(t)$$

Proof (1) when $u > a$ or $u < b$, according to definition (3), obviously $H(u,a,b) = 0$.

(2) when $b \le u \le a$, according to definition (3), we have

$$\begin{aligned}
H(u,a,b) &= P\{\inf_{0\le t<T} U_\delta(t) \ge b \sup_{0\le t<T} U_\delta(t) \le a, T < \infty\} \\
&= \sum_{n=1}^{\infty} P\{\inf_{0\le t<T} U_\delta(t) \ge b \sup_{0\le t<T} U_\delta(t) \le a, T = T_n\} \\
&= \sum_{n=1}^{\infty} P\{b \le U_\delta(T_1) \le a, b \le U_\delta(T_2) \le a, \\
&\qquad \cdots, b \le U_\delta(T_{n-1}) \le a, U_\delta(T_n) < 0\} \\
&= \sum_{n=1}^{\infty} h_n(u,a,b) \qquad (4)
\end{aligned}$$

where

$$h_n(u,a,b) = P\{b \le U_\delta(T_1) \le a,\ b \le U_\delta(T_2) \le a, \cdots, b \le U_\delta(T_{n-1}) \le a, U_\delta(T_n) < 0\}$$

According to the definition, we have

$$\begin{aligned}
h_1(u,a,b) &= P\{U_\delta(T_1) < 0\} \\
&= P\{ue^{\delta T_1} - Y_1 < 0\} = \int_0^{\infty} \overline{F}(ue^{\delta t} + c\overline{S}_{\bar{t}|})dK(t) \qquad (5)
\end{aligned}$$

$$\begin{aligned}
h_2(u,a,b) &= P\{b \le U_\delta(T_1) \le a, U_\delta(T_2) < 0\} \\
&= P\{b \le ue^{\delta T_1} - Y_1 \le a, ue^{\delta T_2} - e^{\delta(T_2 - T_1)}Y_1 - Y_2 < 0\} \\
&= \int_0^{\infty}\int_0^{\infty} P\{b \le ue^{\delta W_1} - Y_1 \le a, ue^{\delta(W_1+W_2)} \\
&\qquad -e^{\delta W_2}Y_1 - Y_2 < 0 \mid Y_1 = y, W_1 = t\}dG(y,t) \\
&= \int_0^{\infty}\int_{ue^{\delta t}-a}^{ue^{\delta t}-b} P\{(ue^{\delta t} - y)e^{\delta W_2} - Y_2 < 0\}dF(y + c\overline{S}_{\bar{t}|})dK(t) \\
&= \int_0^{\infty}\int_{ue^{\delta t}+c\overline{S}_{\bar{t}|}-a}^{ue^{\delta t}+c\overline{S}_{\bar{t}|}-b} P\{(ue^{\delta t} + c\overline{S}_{\bar{t}|} - y)e^{\delta W_2} - Y_2 < 0\}dF(y)dK(t) \\
&= \int_0^{\infty}\int_{ue^{\delta t}+c\overline{S}_{\bar{t}|}-a}^{ue^{\delta t}+c\overline{S}_{\bar{t}|}-b} h_1(ue^{\delta t} + c\overline{S}_{\bar{t}|} - y, a, b)dF(y)dK(t) \qquad (6)
\end{aligned}$$

By inductive assumption, when $n \ge 2$,

$$\begin{aligned}
h_n(u,a,b) &= P\{b \le U_\delta(T_1) \le a, \\
&\qquad , b \le U_\delta(T_2) \le a, \cdots, b \le U_\delta(T_{n-1}) \le a, U_\delta(T_n) < 0\} \\
&= P\{b \le ue^{\delta W_1} - Y_1 \le a, b \le ue^{\delta(W_1+W_2)} - e^{\delta W_2}Y_1 - Y_2 \le a, \cdots, \\
&\qquad b \le ue^{\delta\sum_{j=1}^{n-1}W_j} - \sum_{i=1}^{n-1} Y_i e^{\delta\sum_{j=i+1}^{n-1}W_j} \le a, ue^{\delta\sum_{j=1}^{n}W_j} - \sum_{i=1}^{n} Y_i e^{\delta\sum_{j=i+1}^{n}W_j} < 0\} \\
&= \int_0^{\infty}\int_{ue^{\delta t}-a}^{ue^{\delta t}-b} P\{b \le (ue^{\delta t} - y)e^{\delta W_2} - Y_2 \le a,
\end{aligned}$$

$$\cdots, b \leq (ue^{\delta t} - y)e^{\delta\sum_{j=2}^{n-1} W_j} - \sum_{i=2}^{n-1} Y_i e^{\delta\sum_{j=i+1}^{n-1} W_j} \leq a,$$

$$(ue^{\delta t} - y)e^{\delta\sum_{j=2}^{n} W_j} - \sum_{i=2}^{n} Y_i e^{\delta\sum_{j=i+1}^{n} W_j} < 0\} dG(y,t)$$

$$= \int_0^\infty \int_{ue^{\delta t}-a}^{ue^{\delta t}-b} h_{n-1}(ue^{\delta t} - y, a, b) dF(y + c\bar{S}_{\bar{t}|}) dK(t)$$

$$= \int_0^\infty \int_{ue^{\delta t}+c\bar{S}_{\bar{t}|}-a}^{ue^{\delta t}+c\bar{S}_{\bar{t}|}-b} h_{n-1}(ue^{\delta t} + c\bar{S}_{\bar{t}|} - y, a, b) dF(y) dK(t) \quad (7)$$

Theorem 2 when $b \leq u \leq a$, then $H(u,a,b)$ has the following integral equation

$$H(u,a,b) = \int_0^\infty \bar{F}(ue^{\delta t} + c\bar{S}_{\bar{t}|}) dK(t) + \int_0^\infty \int_{ue^{\delta t}+c\bar{S}_{\bar{t}|}-a}^{ue^{\delta t}+c\bar{S}_{\bar{t}|}-b} H(ue^{\delta t} + c\bar{S}_{\bar{t}|} - y, x) dF(y) dK(t)$$

Proof

$$H(u,a,b) = \sum_{n=1}^\infty h_n(u,a,b) = h_1(u,a,b) + \sum_{n=2}^\infty h_n(u,a,b)$$

$$= h_1(u,a,b) + \sum_{n=2}^\infty \int_0^\infty \int_{ue^{\delta t}+c\bar{S}_{\bar{t}|}-a}^{ue^{\delta t}+c\bar{S}_{\bar{t}|}-b} h_{n-1}(ue^{\delta t} + c\bar{S}_{\bar{t}|} - y, a, b) dF(y) dK(t)$$

$$= h_1(u,a,b) + \int_0^\infty \int_{ue^{\delta t}+c\bar{S}_{\bar{t}|}-a}^{ue^{\delta t}+c\bar{S}_{\bar{t}|}-b} \sum_{n=2}^\infty h_{n-1}(ue^{\delta t} + c\bar{S}_{\bar{t}|} - y, a, b) dF(y) dK(t)$$

$$= \int_0^\infty \bar{F}(ue^{\delta t} + c\bar{S}_{\bar{t}|}) dK(t) + \int_0^\infty \int_{ue^{\delta t}+c\bar{S}_{\bar{t}|}-a}^{ue^{\delta t}+c\bar{S}_{\bar{t}|}-b} H(ue^{\delta t} + c\bar{S}_t - y, a, b) dF(y) dK(t) \quad (8)$$

Corollary 1 In Theorem 1, let $b \to 0^+$, we can get the distribution function of the maximum surplus before the ruin $H(u,a)$

（1） when $a < u$, then $H(u,a) = 0$

（2） when $a \geq u$， then $H(u,a) = \sum_{n=1}^\infty h_n(u,a)$, where $h_1(u,a) = \int_0^\infty \bar{F}(ue^{\delta t} + c\bar{S}_{\bar{t}|}) dK(t)$

When $n \geq 2$

$$h_n(u,a) = \int_0^\infty \int_{ue^{\delta t}+c\bar{S}_{\bar{t}|}-a}^{ue^{\delta t}+c\bar{S}_{\bar{t}|}} h_{n-1}(ue^{\delta t} + c\bar{S}_{\bar{t}|} - y, a) dF(y) dK(t)$$

Proof

let

$$b \to 0^+$$

in Theorem 1, then we have Corollary 1.

Corollary 2 In Theorem 2, let $b \to 0^+$, then $H(u,a)$ has the following integral equation

$$H(u,a) = \int_0^\infty \bar{F}(ue^{\delta t} + c\bar{S}_{\bar{t}|}) dK(t) + \int_0^\infty \int_{ue^{\delta t}+c\bar{S}_{\bar{t}|}-a}^{ue^{\delta t}+c\bar{S}_{\bar{t}|}} H(ue^{\delta t} + c\bar{S}_{\bar{t}|} - y, a) dF(y) dK(t)$$

Proof

let

$$b \to 0^+$$

in Theorem 2, then we have Corollary 2.

Corollary 3 In Theorem 1, let $a \to +\infty$, we can get the distribution function of the minimum surplus before the ruin $H(u,b)$

（1） when $b > u$, then $H(u,b) = 0$

（2） when $b \le u$， then $H(u,b) = \sum_{n=1}^{\infty} h_n(u,b)$,

where

$$h_1(u,b) = \int_0^{\infty} \overline{F}(ue^{\delta t} + c\overline{S}_{\bar{t}|})dK(t)$$

When $n \ge 2$

$$h_n(u,b) = \int_0^{\infty} \int_{ue^{\delta t}+c\overline{S}_{\bar{t}|}}^{ue^{\delta t}+c\overline{S}_{\bar{t}|}-b} h_{n-1}(ue^{\delta t} + c\overline{S}_{\bar{t}|} - y, b)dF(y)dK(t)$$

Proof let $a \to +\infty$ in Theorem 1, then we have Corollary 3.

Corollary 4 In Theorem 2, let $b \to 0^+$, then $H(u,a)$ has the following integral equation

$$K(u,b) = \int_0^{\infty} \overline{F}(ue^{\delta t} + c\overline{S}_{\bar{t}|})dK(t) + \int_0^{\infty} \int_0^{ue^{\delta t}+c\overline{S}_{\bar{t}|}-b} K(ue^{\delta t} + c\overline{S}_{\bar{t}|} - y, b)dF(y)dK(t)$$

Proof let $a \to +\infty$ in Theorem 2, then we have Corollary 4.

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