Dynamical Hamilton Density of Gravitational Field

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Abstract. Following the recent study about "general transverse gauge condition" for gravity in weak field approximation in Ref [5], we revisit the energy density gauge-dependent problem of gravitational field by canonical field theory procedure in this paper. Firstly, we proof the general transverse gauge condition is meaningful and operational. Then by using the general transverse gauge condition, we can get the gauge-invariant solution of Einstein equation of general relativity. The final total Hamilton density of gravitational field obtained by our procedure shows some interesting properties: i) it is absolutely gauge-invariant and thus has no gauge-dependent problem. ii) It can be divided into two parts, pure dynamical and pure instantaneous. Just like what we had known the electromagnetic energy density in electrodynamics. iii) It differs from the traditional effective stress-energy tensor $t_{00} = \frac{1}{2} < \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} >$ (TT means Transverse-Traceless part of metric $h_{\mu\nu}$ in weak field approximation) in radiation by an additional term $h_{jk,i}^{TT} h_{jk,i}^{TT}$, which may contribute some energy in total radiation system.

Introduction

Hamiltonian, which plays the important role in constructing the framework of modern physics, is the total energy of system. If one finds out the Hamiltonian of system, it means one can get all the desired properties of system by neither canonical equations of Hamilton in classical mechanics, nor Schrödinger equation in quantum mechanics. Unfortunately, when considering the interaction system, like vector gauge field A^{μ} couples to external electric fluid source j^{μ}, the Hamiltonian must be gauge-dependent since it is constructed by A^{μ}, which has the so-called gauge degrees of freedom. This is because although gauge theory is a powerful tool to help us to understand the fundamental interaction in nature, the term gauge refers to redundant degrees of freedom in the Lagrangian. The most familiar example is the case of electromagnetic interaction. Since gauge invariance principle tells us that Lagrangian of Dirac electric field must be invariant under $U(1) = e^{i\omega(x)}$ transformation ($\omega(x)$ is an arbitrary scalar function), in order to keep this invariant property of Lagrangian, the ordinary differential ∂_{μ} should be replaced by covariant differential $D_{\mu} = \partial_{\mu} - iqA_{\mu}$. It results two conclusions, on the one hand, the four-dimensional A^{μ} is naturally appeared in the total Lagrangian and thus the electromagnetic interaction is presented. One the other hand, the gauge theory requires A^{μ} transforms under $U(1)$ as $A^{\mu} \to A'^{\mu} = A^{\mu} - \partial^{\mu}\omega$. It is easy to check that both A^{μ} and A'^{μ} give the same field strength $F^{\mu\nu} = \partial_{\mu}A^{\nu} - \partial_{\nu}A^{\mu} \to F'^{\mu\nu} = F^{\mu\nu}$. This means one can't distinguish A^{μ} and A'^{μ} by experiment. On the other word, the arbitrary scalar function $\omega(x)$ brings in one gauge degree of freedom which must be removed when identifying real physical massless photon with two physical polarizations. The usually technique to remove the above gauge degree of freedom is introducing one gauge fixing condition by hand . Several gauge fixings, like Lorentz gauge, Coulomb (transverse or radiation) gauge, are most high appearance rate in textbooks or literature. But only Coulomb gauge $\vec{\partial}\cdot\vec{A}=0$ can pick out the two physical (and at the same time dynamical) components $\vec{A}_{\perp}$. So that vector field can be divided into transverse and longitudinal parts, $\vec{A}\equiv\vec{A}_{\perp}+\vec{A}_{\parallel}$, defined by $\vec{\partial}\cdot\vec{A}_{\perp}=0$ and $\vec{\partial}\times\vec{A}_{\parallel}=0$, and a real photon is described by transverse field $\vec{A}_{\perp}$. Now, one can use the real physical part $\vec{A}_{\perp}$ to construct the Hamiltonian which is absolutely gauge-invariant and luckily also dynamical.

Motivated by the dynamical Hamiltonian of vector gauge field A^{μ} can be perfectly solved via finding one pertinent gauge fixing, we will discuss the dynamical Hamiltonian of gravitational field (GF) in this paper. First of all, we will recall what we had done about gauge properties of GF when using gauge-invariance principle to gravitational field in second Section. Specifically, we find the pertinent general transverse gauge fixing for gravity which can remove the non-physical degrees of freedom of metric and affine connection. Interestingly, this gauge fixing is two parameters (ab) related, or in other words, it is not unique. Then, we discuss the solution of Einstein equation in the above general transverse gauge fixing in third Section. We find the final gauge-invariant and dynamical metric is also dependent on the choice of parameters in general transverse gauge fixing. Then following the canonical field theory procedure, we use the gauge-invariant metric tensor to construct Hamilton density of GF in fourth Section. We find that the final Hamilton density of GF can be divided into dynamical and non-dynamical parts. final Section is the summarization of the paper. Throughout this paper, an over dot denotes time derivative, Greek indices run from 0 to 3, Latin indices run from 1 to 3, and repeated indices are summed over even when two spatial indices are both upstairs or downstairs, ∂ or comma denotes ordinary derivative.

General Transverse Gauge Condition of Gravitational Field

Einstein equivalence principle indicates that gravitational effect is described by the metric tensor $g_{\mu\nu}$, which characterizes the inertial effect associated with coordinate choice. But sometimes, one does face the necessity of identifying the real gravitational degrees of freedom out of the metric, e.g. , in associating a meaningful energy to gravitational radiation, in analyzing canonical structure of gravitation and thus quantizing it to define a physical graviton. Thus, how to separate inertial effect out of true physical gravitational field is well motivated in physics. Such separation can be concluded as various physical decomposition of gravitational field in Refs. [1-3]. If regarding gravitational field as one of gauge field [4], those physical decompositions are equivalent to finding some gauge fixing conditions for gravity [5].

In general relativity, gauge-invariance principle refers to general covariance under arbitrary coordinate transformation. The number of gauge degrees of freedom of gravitational field is four. Following the hints from the vector field, finding a complete gauge constraints on $h_{\mu\nu}$ which can pick out real physical components in full metric is an important work for constructing gauge-invariant energy density of pure gravitational field.

In this paper, we will focus on linearized general relativity or weak-field approximation, which means the ordinary metric tensor can be written as $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, here $\eta_{\mu\nu}$ presents Minkowski metric and $h_{\mu\nu}\ll 1$ presents weak curvature of space-time. We can show that such a complete gauge condition is

$$\partial_i h_{0i}+a\partial_0 h_{ii}=0,\partial_i h_{ji}+b\partial_j h_{ii}=0 \tag{1}$$

The parameters a and b can be any value except $b=-1$. The proof is divided into two-fold(the full discussions about gauge Eq. (1) can be seen in Ref. [5]):

1) Eq. (1) permits no more gauge freedom.

Proof: The transformation property of $h_{\mu\nu}$ under linear gauge transformation reads:

$$h_{\mu\nu}(x) \to h'_{\mu\nu}(x) = h_{\mu\nu}(x) + \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) \tag{2}$$

where $\xi_\mu(x)$ are four arbitrary gauge parameters. Obviously the gauge freedoms of GF are four, and the constraint of $h_{\mu\nu}$ through Eq. (1) is just four.

2) the gauge-transformation parameter $\xi_\mu(x)$ that brings $h_{\mu\nu}$ to the gauge Eq. (1) is unique.

Proof: If $\xi_\mu(x)$ brings $h_{\mu\nu}$ to the gauge Eq. (1) is unique which means the below

$$\partial_i h'_{i0} + a\partial_0 h'_{ii} = 0, \partial_i h'_{ij} + b\partial_j h'_{ii} = 0 \tag{3}$$

We cast Eqs. (3) into the equations for $\xi_\mu(x)$:

$$\vec{\partial}^2 \xi_0 + (1+2a)\partial_0 \partial_i \xi_i = \partial_i h_{i0} + a\partial_0 h_{ii}\,,\quad \vec{\partial}^2 \xi_j + (1+2b)\partial_j \partial_i \xi_i = \partial_i h_{ij} + b\partial_j h_{ii} \tag{4}$$

To solve, act on both sides of Eq. (4b) with ∂_j and sum over j, we can get

$$\xi_{i,i} = \frac{1}{2(1+b)}(bh_{ii} + \frac{1}{\vec{\partial}^2} h_{ij,ij}) \tag{5}$$

It is easy to see that why is $b=-1$ excluded since $\partial_i h_{ji} - \partial_j h_{ii}$ has a gauge-invariant divergence and thus is unable to fix any gauge. We remind that inversion of the Laplacian operator $\vec{\partial}^2$ implies a vanishing boundary value at infinity. Substituting $\xi_{i,i}$ back into Eqs. (4)

$$\xi_0 = \frac{1}{\vec{\partial}^2}[h_{0i,i} + \frac{2a-b}{2(1+b)} h_{ii,0} - \frac{1+2a}{2(1+b)} \frac{1}{\vec{\partial}^2} h_{ik,ik0}] \tag{6a}$$

$$\xi_j = \frac{1}{\vec{\partial}^2}[h_{ij,i} + \frac{b}{2(1+b)} h_{ii,j} - \frac{1+2b}{2(1+b)} \frac{1}{\vec{\partial}^2} h_{ik,ikj}] \tag{6b}$$

These are the unique ξ_μ that bring $h'_{\mu\nu}$ into the gauge Eq. (1).

The Solution of Einstein Equation in General Transverse Gauge

Annoyingly, the freedom of choices of parameter a,b in Eq. (1) shows that the general transverse gauge is not unique. When casting gauge Eq. (1) into linearized Einstein's field equation below

$$\Box h_{\mu\nu} - \partial_\mu \partial_\rho h^\rho_\nu - \partial_\nu \partial_\rho h^\rho_\mu + \partial_\mu \partial_\nu h^\rho_\rho = -S_{\mu\nu} \tag{7}$$

Here $\Box = \partial^2 - \partial_t^2$, $S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^\rho_\rho$, and we put $16\pi G = 1$. we can find that different choices of a,b will bring different difficulties to solve solution of field equation. Then casting gauge Eq.(1) to Eq.(7), we can finally get the solution of linearized Einstein's field equation in general transverse gauge Eq.(1) after tedious calculation

$$h^{(ab)}_{00} = -\frac{1}{\vec{\partial}^2} S_{00} + \frac{1+2a}{1+b}\frac{1}{\vec{\partial}^2}\frac{1}{\vec{\partial}^2} T_{00,00}, h^{(ab)}_{0i} = -\frac{1}{\vec{\partial}^2} S_{0i} + \frac{1+a+b}{1+b}\frac{1}{\vec{\partial}^2}\frac{1}{\vec{\partial}^2} T_{00,0i} \tag{8a}$$

$$h^{(ab)}_{ii} = -\frac{1}{1+b}\frac{1}{\vec{\partial}^2} T_{00}, h^{(b)}_{ji,i} = \frac{b}{1+b}\frac{1}{\vec{\partial}^2} T_{00,j}, \Box h^{(b)}_{ij} = -\hat{S}^{(b)}_{ij} \tag{8b}$$

The source term $\hat{S}^{(b)}_{ij}$ is:

$$\hat{S}^{(b)}_{ij} = S_{ij} - \frac{1}{\vec{\partial}^2}(S_{ki,kj} + S_{kj,ki} - S_{kk,ij}) - \frac{1+2b}{1+b}\frac{1}{\vec{\partial}^2}(S_{kk} - \frac{1}{\vec{\partial}^2} S_{kl,kl})_{,ij} \tag{9}$$

We can see that, although any choice of a,b can remove all non-physical degrees of freedom of $h_{\mu\nu}$, some special choices of a,b like $a=b=-\frac{1}{2}$ which kills the double non-localized operator $\frac{1}{\vec{\partial}^2}\frac{1}{\vec{\partial}^2}$ can simplify solution Eq. (8) to the maximum extent [7,8]. On the other hand, there is always existed a useful gauge, which also claims can remove all gauge freedom of gravitational field, is transverse-traceless (TT) gauge for long time (first proposed by Arnowitt, Deser and Misner (ADM) [2]) and also be discussed in Ref. [5]).

$$h_{0\mu}=0,\partial_i h^i_j=0,h^i_i=0 \tag{10}$$

TT gauge Eq. (10) has altogether eight independent constraints. Obviously they are too harsh restrictions, since the gauge freedom of metric tensor $h_{\mu\nu}$ as we had shown above is only four. But if the metric tensor $h_{\mu\nu}$ is given by any technique, it is a conventional usage to use TT component of h_{ij} (h^{TT}_{ij}) to construct pure physical (or dynamical) quantity about gravitational field (like stress-energy tensor for gravitational wave [4]

$$h^{TT}_{ij}=h_{ij}-\frac{1}{2}\delta_{ij}(h_{kk}-\frac{1}{\vec{\partial}^2}\partial_k\partial_l h_{kl})-\frac{1}{\vec{\partial}^2}(\partial_k\partial_j h_{ik}+\partial_k\partial_i h_{jk}-\frac{1}{2}\partial_i\partial_j h_{kk}-\frac{1}{2}\frac{1}{\vec{\partial}^2}\partial_k\partial_l\partial_i\partial_j h_{kl}) \tag{11}$$

As we know the metric is always solved by Einstein field equation combine with gauge fixing. The motivation of finding how TT components h^{TT}_{ij} are related to metric tensor h_{ij} is intuitive. It is amazing that the solution of Einstein field equation in general transverse gauge $h^{(ab)}_{\mu\nu}$ can be associated with the TT gauge by the following formula[5,6]:

$$h^{(ab)}_{ij}=h^{TT}_{ij}+\frac{1+b}{2}\delta_{ij}h^{(ab)}-\frac{1+3b}{2}\frac{1}{\vec{\partial}^2}\partial_i\partial_j h^{(ab)} \tag{12}$$

Pure Physical Dynamical Hamilton Density of Gravitational Field

We are now in the position to use gauge-invariant metric $h^{ab}_{\mu\nu}$ (Eq. (8)) and h^{TT}_{ij} (Eq. (12)) to construct gauge-invariant and dynamical Hamilton of gravitational field. The Lagrangian density of gravitational field couples to an external conserved source $T^{\mu\nu}(x)$ which is invariant under gauge transformation Eq. (2) is:

$$L=\frac{1}{4}(\partial_\mu h^\alpha_\alpha\partial^\mu h^\beta_\beta-\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta}+2\partial_\mu h^{\mu\alpha}\partial^\nu h_{\nu\alpha}-2\partial_\mu h^\alpha_\alpha\partial_\nu h^{\mu\nu})+\frac{\kappa}{2}h_{\mu\nu}T^{\mu\nu} \tag{13}$$

Given the Lagrangian, we can proceed to construct the canonical Hamilton of gravitational field

$$H=\Pi_{\mu\nu}\dot{h}^{\mu\nu}-L \tag{14}$$

Here $\frac{\partial L}{\partial\dot{h}^{\mu\nu}}=\Pi_{\mu\nu}$ is the momentum conjugate of $h^{\mu\nu}$. The momentum conjugates are

$$\Pi_{00}=\frac{1}{2}\partial_i h_{0i},\Pi_{0i}=\frac{1}{2}(\partial_i h_{00}-\partial_i h+2\partial_j h_{ij}),\Pi_{ij}=\frac{1}{2}[\dot{h}_{ij}+\delta_{ij}(\partial_k h_{0k}-\dot{h})] \tag{15}$$

Here $h\equiv h_{ii}$ is the spatial trace. (we have identified $\dot{h}_{0i}$ with $\dot{h}_{i0}$, but not $\dot{h}_{ij}$ with $\dot{h}_{ji}$, thus we are going to sum over both (ij) and (ji), but not $(i0)$.). It is now straightforward to calculate the Hamilton quantity H: (Note that H is the Hamilton quantity and H is the Hamiltonian density)

$$\mathrm{H}=\int d^3x(\Pi_{\mu\nu}\dot{h}^{\mu\nu}-L)=\int d^3x(\Pi_{00}\dot{h}^{00}+\Pi_{ij}\dot{h}^{ij}+\Pi_{0i}\dot{h}^{0i}-L) \tag{16}$$

When casting Eq. (1) and Eq.(12) into the equations for Eq. (16), the final gauge-invariant canonical Hamilton of GF can be written by two parts:

$$\mathrm{H}^{(ab)} = \mathrm{H}_{dyn} + \mathrm{H}_{int} \tag{17}$$

$$\mathrm{H}_{dyn} = \frac{1}{4}\int d^3x(\dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT} + \partial_i h_{jk}^{TT}\partial_i h_{jk}^{TT}) \tag{18a}$$

$$\mathrm{H}_{int} = \frac{1}{4}\int d^3x(2h_{ij}^{TT}T_{ij} + \frac{1}{2}T_{00}\frac{1}{\vec{\partial}^2}T_{00} + T_{00}\frac{1}{\vec{\partial}^2}T_{kk} - 2T_{0i}\frac{1}{\vec{\partial}^2}T_{0i} + \frac{5b-8a-3}{2(1+b)}\frac{1}{\vec{\partial}^2}\dot{T}_{00}\frac{1}{\vec{\partial}^2}\dot{T}_{00}) \tag{18b}$$

Here the superscript (ab) means the calculation are doing in the general transverse gauge condition Eq. (1), and the Eq. (18a) and Eq. (18b) are the gauge-invariant Hamilton quantity of gravitational field. Specifically, Eq. (18a) is the dynamical and gauge-invariant Hamilton quantity and Eq. (18b), although gauge-invariant, is non-dynamical and the instantaneous effect of gravitational field. Like the Coulomb potential of electrostatic field, the instantaneous effect of gravitational field is not radioactive.

Summary

From the above discussions, we can find that, by using the general transverse gauge condition, Eq. (1), for gravitational field, the gauge-invariant Hamilton density can be reasonable constructed, Eq. (18a) and Eq. (18b). The difference between ``dynamical'' and ``non-dynamical''. E. g., in electrodynamics, the instantaneous Coulomb potential is non-dynamical but physical (contributed by A^0), and must be included into the total Hamiltonian of the system. Similarly, for gravity the instantaneous Newtonian interaction (contributed by $h_{0\mu}$ and h_{ii}) is non-dynamical but physical, and must be included into the total Hamiltonian, especially in a quantum theory [9]. Finally, now that Eq. (18a) can be explained as dynamical Hamilton density of GF, it is necessary to recall previous similar work. The so-called stress-energy tensor for gravitational waves in Ref. [10] is always used to be powerful tool to calculate energy of gravitational wave[10]:

$$T_{00}^{GW} = \frac{1}{2} < \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT} > \tag{19}$$

Here superscripts GW mean gravitational wave, and brackets <> denote an average over several wavelengths. The only difference between Eq. (18a) and Eq. (19) is the peculiar redundant term $\partial_i h_{jk}^{TT}\partial_i h_{jk}^{TT}$ in Eq. (18a). This difference shows our work is interesting and useful. One can use the Eq. (18a) to calculate the various concrete energy density of gravitational source .

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