Logarithmic Normal Population Distribution of Parameter of Joint Fiducial Region Estimation

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Abstract. According to the lognormal distribution and normal distribution relationship of logarithmic normal distribution is given two parameters interval estimation of two lognormal distribution parameters and Combination fiducial region estimation.

The Definition of Lognormal Distribution

Lognormal distribution is named because of the logarithmic obey normal distribution, the so called lognormal distribution. Definition: a population X obey the lognormal distribution, the distribution density function is

$$f(x;\mu,\sigma^2)=\begin{cases}\dfrac{1}{\sqrt{2\pi}\sigma x}e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} & x>0\\ 0 & x\le 0\end{cases} \tag{1}$$

Among them $-\infty<\mu<\infty,\ \sigma^2>0$

Establish Function Model, the Induction of Faith Distribution

Definition**: se**t to simple random sample $X_1, X_2, ...X_n$ from lognormal population X, to remember

$$\overline{\ln X} = \frac{1}{n}\sum_{i=1}^{n}\ln X_i$$

$$Q^2 = \sum_{i=1}^{n}(\ln X_i - \overline{\ln X})^2$$

According to the lognormal distribution and normal distribution and the relationship between basic theorem, the sampling distribution has the following lemma:

Lemma 1.

(1) $\overline{\ln X}$ 与 Q^2 independent of each other

(2) $\sqrt{n(n-1)}\dfrac{\overline{\ln X}-\mu}{Q}$ ~ t (n-1) (2)

(3) $\dfrac{Q^2}{\sigma^2}$ ~ χ^2(n-1)

(4) ($\overline{\ln X}$, Q^2) is a sufficient statistic(μ,σ^2)

Due to the sufficient statistic of ($\overline{\ln X}$, Q^2) is ($\overline{\ln X}$, Q^2),So we based on the principle of sufficiency, consider the combination of (μ,σ^2) by ($\overline{\ln X}$, Q^2) combination fiducial region estimation .Because $\overline{\ln X}$ ~ N($\mu, \frac{\sigma^2}{n}$), By the lemma (1.1) and (3) to know $\frac{Q^2}{\sigma^2}$ ~ χ^2 (n-1), Again by lemma (1.1) and (1) $\overline{\ln X}$ 与 Q^2 independent of each other. If remember the error variables e_1 ~ N(0,1), e_2 ~ χ^2 (n-1),and e_1 , e_2 are independent of each other. The function model is set up

$$\begin{cases} \overline{\ln X} = \mu + \frac{\sigma}{\sqrt{n}} \cdot e_1 \\ Q^2 = \sigma^2 \cdot e_2 \end{cases} \tag{3}$$

One ($\overline{\ln X}$, Q^2) as the sample observations;(μ, σ^2) for the unknown parameters; (e_1 , e_2) as the error variable. Transposition function model by on (3)

This can be the joint distribution of faith of (μ, σ^2). In a sample observation value X, thus has the $\overline{\ln X}$ and Q^2 , induced by function model the combination fiducial distribution of (μ, σ^2). Due to the e_1 ~ N(0,1), e_2 ~ χ^2 (n-1), and e_1 and 与 e_2 are independent of each other. Therefore the joint distribution density function of (e_1 , e_2)is

h(e_1 , e_2)= $\frac{1}{\sqrt{2\pi}} e^{-\frac{e_1^2}{2}} \cdot \frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} e_2^{\frac{n-1}{2}-1} \cdot e^{-\frac{e_2}{2}}$ among them $-\infty < e_1 < \infty$, $e_2 > 0$,

Due to $e_1 = \frac{\sqrt{n(\overline{\ln x} - \mu)}}{\sigma}$, $e_2 = \frac{Q^2}{\sigma^2}$ according to random vector transform formula: from

(e_1 , e_2) to (μ, σ^2) transformation, the transformation of the absolute value of the jacobian for

$$|J| = \left\| \begin{matrix} \frac{\partial e_1}{\partial \mu} & \frac{\partial e_1}{\partial \sigma^2} \\ \frac{\partial e_2}{\partial \mu} & \frac{\partial e_2}{\partial \sigma^2} \end{matrix} \right\| = \left\| \begin{matrix} \frac{-\sqrt{n}}{\sigma} & \frac{-1}{2}\sqrt{n}(\overline{\ln x} - \mu)(\sigma^2)^{-\frac{3}{2}} \\ 0 & -Q^2 \frac{1}{\sigma^4} \end{matrix} \right\| = Q^2 \frac{\sqrt{n}}{\sigma^5} \tag{4}$$

In e_1 , e_2 available (μ, σ^2) combination fiducial distribution density function is

$$g(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{n(\overline{\ln x} - \mu)^2}{2\sigma^2}} \cdot \frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} (\frac{Q^2}{\sigma^2})^{\frac{n-3}{2}} \cdot e^{-\frac{Q^2}{2\sigma^2}} Q^2 \frac{\sqrt{n}}{\sigma^5}$$

$$= \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \exp[-\frac{n(\overline{\ln x} - \mu)^2}{2\sigma^2}] \frac{(Q_2)^{\frac{n-1}{2}}}{(2\sigma^2)^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} \cdot e^{-\frac{Q^2}{2\sigma^2}} \tag{5}$$

(5) compared with (7), it is very similar. (3.5) from 0 to ∞ to σ^2 points, can be marginal beliefs about μ distribution; (3.5) from $-\infty$ to ∞ to μ points, can be on σ^2 the marginal distribution of religion, which is available in principle.

We also can use another way: we also can be directly by (1.3), respectively, the marginal fiducial of μ and σ^2 distribution is deduced. μ is derived under the marginal fiducial distribution:

By (3) and the definition of the t distribution we can derive

$$\sqrt{(n-1)} \frac{\sqrt{n}(\mu - \overline{\ln X})}{Q} = -\sqrt{n-1} \cdot \frac{e_1}{\sqrt{e_2}} \sim t\,(n-1) \tag{6}$$

Therefore the marginal fiducial distribution of μ is t distribution.

Under the marginal fiducia distribution of σ^2 is derived:

By (3) can also $\frac{\sigma^2}{Q^2}=\frac{1}{e_2}$, $e_2 \sim \chi^2$ (n-1)

So the marginal fiducial distribution of σ^2 is reversal χ^2 distribution, the distribution density function is

$$\propto \exp\{-\frac{Q^2}{2\sigma^2}\}(\sigma^2)^{-\frac{n+1}{2}} \qquad \sigma^2>0 \tag{7}$$

Fiducial Region (interval) Estimates Is Given

To solve the fiducial interval of μ:

Fiducial distribution of μ Given by (6) for μ interval estimation for fiducial level $1-\alpha$ is

$$[\overline{\ln X} - \frac{s}{\sqrt{n}}t_{1-\frac{\alpha}{2}}(n-1), \overline{\ln X} + \frac{s}{\sqrt{n}}t_{1-\frac{\alpha}{2}}(n-1)]$$

Because the t distribution is unimodal symmetric distribution, so the fiducial interval length is the shortest, is an optimal fiducial interval.

To solve the fiducial interval of σ^2:

$\chi^2_{\frac{\alpha}{2}}(n-1) \le e_2 \le \chi^2_{1-\frac{\alpha}{2}}(n-1)$ Inequality, available σ^2 for the $1-\alpha$ fiducial level fiducial interval $[\frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2}}(n-1)}, \frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2}}(n-1)}]$(Note: this is not the optimum interval estimation.)

All available the following theorem

Theorem 1.1: set X to lognormal population distribution N(μ, σ^2), the distribution density function is

$$f(x;\mu,\sigma^2)=\begin{cases}\frac{1}{\sqrt{2\pi}\sigma x}e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} & x>0\\ 0 & x\le 0\end{cases}$$

among them $-\infty<\mu<\infty$, $\sigma^2>0$ is parameter, then the combination fiducial region of fiducial level $1-\alpha$ for (μ, σ^2)is

$$G=\left\{(\mu,\sigma^2)\,|\,(\overline{\ln X}-\mu)^2 \le \frac{\sigma^2}{n}a_2{}^2, \quad \frac{Q^2}{b_2}<\sigma^2<\frac{Q^2}{b_1}\right\}$$

The calculation formula for the area of the region G is

$$S_G = 2\int_{\frac{(n-1)s^2}{b_2}}^{\frac{(n-1)s^2}{b_1}}\int_{\ln x}^{\frac{\sigma}{\sqrt{n}}a_2} d\mu d\sigma^2 = 2\int_{\frac{(n-1)s^2}{b_2}}^{\frac{(n-1)s^2}{b_1}}\frac{\sigma}{\sqrt{n}}a_2 d\sigma^2 = \frac{4a_2}{3\sqrt{n}}(n-1)^{\frac{3}{2}}s^3(b_1^{-\frac{3}{2}}-b_2^{-\frac{3}{2}})$$

(For a given level of faith $1-\alpha$, to select suitable a_1, a_2, b_1, b_2.)

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