Strong convergence theorems for k-strictly pseudo-contractive mapping in Hilbert spaces

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Abstract: In this paper, we introduce a general iterative algorithm and prove strong convergence theorems for a non-self k-strictly pseudo-contractive mappings in Hilbert spaces. Our results improve and extend the corresponding results announced by many others.

Introduction and Preliminaries

Let K be a nonempty subset of a Hilbert space H. Recall that a mapping $T:K\to H$ is said to be a k-strictly pseudo-contractive if there exists a constant $k\in[0,1)$ such that

$$\|Tx-Ty\|^2\le\|x-y\|^2+k\|(I-T)x-(I-T)y\|^2 \text{ for all } x,y\in K. \tag{1.1}$$

Note that the class of k-strictly pseudo-contractions includes strictly the class of nonexpansive mapping which are mappings T on K such that

$$\|Tx-Ty\|\le\|x-y\|,\forall x,y\in K. \tag{1.2}$$

That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive.

In 2002, Marino and Xu[1] introduced and considered the following iterative algorithm:

$$\begin{cases} x_0\in K, \\ x_{n+1}=a_n g\,f(x_n)+(I-a_nA)Tx_n,\forall n\ge 0. \end{cases} \tag{1.3}$$

Theorem MX. Let H be a Hilbert space, K be a closed convex subset of H, $T:K\to K$ be a nonexpansive mapping with $F(T)\ne\varnothing$. Let A be a strong positive bounded linear operator on K with coefficient $\bar{g}$ and $f:K\to K$ be a contraction with the contractive coefficient $(0<a<1)$ such that $0<g<\frac{\bar{g}}{a}$. Let $\{x_n\}$ be a sequence in K generated by (1.3). Then, under the hypotheses (i) $\lim_{n\to\infty}a_n=0$, (ii) $\sum_{n=1}^{\infty}a_n=\infty$ and (iii) either $\sum_{n=1}^{\infty}|a_{n+1}-a_n|<\infty$ or $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1$, $\{x_n\}$ converges strongly to a fixed point q of T, which is the unique solution of the following variational inequality related to the linear operator A:

$$\langle(A-g\,f)q,q-p\rangle\le 0,\forall p\in F(T).$$

In this paper, motivated by Marino and Xu, we introduce a general iterative and prove strong convergence theorems for k-strictly pseudo-contractive mappings in Hilbert spaces. Our results improve and extend the corresponding ones announced by many others.

Throughout this paper, we use $F(T)$ to denote the fixed point set of the mapping T and P_K to denote the metric projection of a Hilbert space H onto a closed convex subset K of H. Recall that a self-mapping $f:K\to K$ is a contraction on K if there exists a constant $a\in(0,1)$ such that

$$\|f(x)-f(y)\|\le a\|x-y\|,\forall x,y\in K. \tag{1.4}$$

In order to prove our main results, we need the following definitions and lemmas.

Lemma 1.1[2] If T is a k-strictly pseudo-contraction on a closed convex subset of K of a real Hilbert space H, then the fixed point set $F(T)$ is closed convex so that the projection $P_{F(T)}$ is well defined.

Lemma 1.2[2] Let H be a Hilbert space, K be a closed convex subset of H. Let $T: K \to H$ be a k-strictly pseudo-contractive mapping with $F(T) \neq \varnothing$ .Then $F(P_K T) = F(T)$.

Lemma 1.3[2] Let $T: K \to H$ be a k-strictly pseudo-contraction. Define $S: K \to H$ by $Sx = l\,x + (1-l)Tx$ for each $x \in K$ .Then, as $l \in [k,1)$, S is a nonexpansive mapping such that $F(S) = F(T)$.

Lemma 1.4[3] Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \le (1-g_n)a_n + d_n, \forall n \ge 0.$ where $\{g_n\}$ is a sequence in $(0,1)$ and $\{d_n\}$ is a sequence such that (i) $\sum_{n=1}^{\infty} g_n = \infty$; (ii) $\limsup_{n\to\infty} \frac{d_n}{g_n} \le 0$ or $\sum_{n=1}^{\infty} |d_n| < \infty$, Then $\lim_{n\to\infty} a_n = 0.$

Lemma 1.5[4]**.** Let H be a real Hilbert space, the following inequality holds
$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \forall x, y \in H.$

Main results

Theorem2.1. Let K be a nonempty closed convex subset of a real Hilbert space H and $T: K \to H$ be a k-strictly pseudo-contractive mapping with a common fixed point for some $0 \le k < 1$. Let $f: K \to K$ be a contraction with the contractive coefficient $(0 < a < 1)$. Let $\{x_n\}$ be a sequence in K generated in the following manner:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = a_n f(x_n) + (1-a_n)P_K S x_n, \forall n \ge 1. \end{cases}$$

where $S: K \to H$ is a mapping defined by $Sx = l\,x + (1-l)Tx$ .If the control sequence $\{a_n\}$ satisfies the following conditions: (i) $\lim_{n\to\infty} a_n = 0$; (ii) $\sum_{n=1}^{\infty} a_n = \infty$; (iii) $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$. Then $\{x_n\}$ converges strongly to a fixed point q of T, which solves the following variational inequality:

$$\langle f(q) - q, p - q\rangle \le 0, \forall p \in F(T).$$

Proof. From Lemma1.3, we know that the mapping $S: K \to H$ is a nonexpansive mapping and $F(S) = F(T)$, By our assumptions on T, we have $F(T) \neq \varnothing$. By Lemma1.1, we see $F(P_K S) = F(S) \neq \varnothing$. Since $P_K: H \to K$ is a nonexpansive mapping, we conclude that $P_K S: K \to K$ is also nonexpansive. Observing the condition (i), we may assume that $a_n < 1$ for all $n \ge 1$. Taking a point $p \in F(T)$,
we obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \|a_n(f(x_n) - p) + (1-a_n)(P_K S x_n - p)\| \\ &\le (1-a_n)\|P_K S x_n - p\| + a_n\|f(x_n) - p\| \\ &\le [1 - a_n(1-a)]\|x_n - p\| + a_n\|f(p) - p\|. \end{aligned}$$

By simple inductions, we have $\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|p - f(p)\|}{1-a}\right\}, \forall n \ge 1,$

which yields that the sequence $\{x_n\}$ is bounded. On the other hand, we have

$$\begin{aligned} x_{n+2} - x_{n+1} &= (1-a_n)(P_K S x_{n+1} - P_K S x_n) - (a_{n+1} - a_n)P_K S x_n \\ &\quad + [a_{n+1}(f(x_{n+1}) - f(x_n)) + f(x_n)(a_{n+1} - a_n)]. \end{aligned}$$

which yields that

$$\begin{aligned}\|x_{n+2}-x_{n+1}\| &\le (1-a_{n+1})\|x_{n+1}-x_n\|+|a_{n+1}-a_n|\,\|P_K Sx_n\| \\ &\quad +[a_{n+1}a\,\|x_{n+1}-x_n\|]+\|f(x_n)\|\|a_{n+1}-a_n\| \\ &\le [1-a_{n+1}(1-a)]\|x_{n+1}-x_n\|+|a_{n+1}-a_n|\,M_1, \end{aligned} \tag{2.1}$$

where M_1 is an appropriate constant such that $M_1 \ge \sup_{n\ge 1}\{\|P_K Sx_n\|+\|f(x_{n\partial})\|\}$.

Noticing the condition (i), (ii) and (iii) and apply Lemma (1.4) to (2.1),

we have $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$ (2.2)

Notice that $\|x_n-P_K S_n\| \le \|x_n-x_{n+1}\|+\|x_{n+1}-P_K Sx_n\| \le \|x_{n+1}-x_n\|+a_n\|f(x_n)-P_K Sx_n\|.$

It follows from the condition (i) and (2.2) that $\lim_{n\to\infty}\|x_n-P_K Sx_n\|=0.$ (2.3)

Next we claim that $\limsup_{n\to\infty}\langle f(q)-q, x_n-q\rangle \le 0,$

(2.4)

where $q=\lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction x **a** $tf(x)+(1-t)P_K Sx.$

Then x_t solves the fixed point equation $x_t = tf(x_t)+(1-t)P_K Sx_t$. Thus we have

$$\|x_t-x_n\| = \|(1-t)(P_K Sx_t-x_n)+t(f(x_t)-x_n)\|.$$

It follows from the Lemma 1.5 that

$$\begin{aligned}\|x_t-x_n\|^2 &= \|(1-t)(P_K Sx_t-x_n)+t(f(x_t)-x_n)\|^2 \\ &\le (1-t)^2\|P_K Sx_t-x_n\|^2+2t\langle f(x_t)-x_n, x_t-x_n\rangle \\ &\le (1-2t+t^2)\|x_t-x_n\|^2+f_n(t)+2t\langle f(x_t)-x_t, x_t-x_n\rangle+2t\langle x_t-x_n, x_t-x_n\rangle, \end{aligned} \tag{2.5}$$

where $f_n(t)=(2\|x_t-x_n\|+\|x_n-P_K Sx_n\|)\|x_n-P_K Sx_n\|\to 0\,(n\to\infty).$ (2.6)

and $\langle x_t-x_n, x_t-x_n\rangle = \|x_t-x_n\|^2.$ (2.7)

Combining (2.5) and (2.7), we have

$$\begin{aligned}2t\langle x_t-f(x_t), x_t-x_n\rangle &\le (t^2-2t)\|x_t-x_n\|^2+f_n(t)+2t\langle x_t-x_n, x_t-x_n\rangle \\ &\le t^2\|x_t-x_n\|^2+f_n(t)\end{aligned}$$

It follows that $\langle x_t-f(x_t), x_t-x_n\rangle \le \frac{t}{2}\|x_t-x_n\|^2+\frac{1}{2t}f_n(t).$ (2.8)

Letting $n\to\infty$ in (2.8) and noting (2.6) yields

$$\limsup_{n\to\infty}\langle x_t-f(x_t), x_t-x_n\rangle \le \frac{t}{2}M_2, \tag{2.9}$$

where $M_2>0$ is a constant such that $M_2 \ge \|x_t-x_n\|^2$ for all $t\in(0,1)$ and $n\ge 1$. Taking $t\to 0$ in (2.9), we have $\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t-f(x_t), x_t-x_n\rangle \le 0.$ (2.10)

On the other hand, we have

$$\begin{aligned}\langle f(q)-q, x_n-q\rangle &= \langle f(q)-q, x_n-q\rangle-\langle f(q)-q, x_n-x_t\rangle \\ &\quad +\langle f(q)-q, x_n-x_t\rangle-\langle f(q)-x_t, x_n-x_t\rangle \\ &\quad +\langle f(q)-x_t, x_n-x_t\rangle-\langle f(x_t)-x_t, x_n-x_t\rangle+\langle f(x_t)-x_t, x_n-x_t\rangle.\end{aligned}$$

It follows that

$$\limsup_{n\to\infty}\langle f(q)-q, x_n-q\rangle \le \|f(q)-q\|\|x_t-q\|+\|x_t-q\|\lim_{n\to\infty}\|x_n-x_t\|$$

$$+a\,\|q-x_t\|\lim_{n\to\infty}\|x_n-x_t\|+\limsup_{n\to\infty}\langle f(x_t)-x_t,x_n-x_t\rangle.$$

Therefore, from (2.10), it follows that

$$\limsup_{n\to\infty}\langle f(q)-q,x_n-q\rangle=\limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q)-q,x_n-q\rangle\le 0.$$

Hence (2.4) holds. Now from the Lemma 1.5, we have

$$\begin{aligned}\|x_{n+1}-q\|^2&=\|(1-a_n)(P_KSx_n-q)+a_n(f(x_n)-q)\|^2\\&\le(1-a_n)^2\|x_n-q\|^2+a_na\left(\|x_n-q\|^2+\|x_{n+1}-q\|^2\right)+2a_n\langle f(q)-q,x_{n+1}-q\rangle,\end{aligned}\tag{2.11}$$

which implies that

$$\|x_{n+1}-q\|^2\le\frac{(1-a_n)^2+a_na}{1-a_na}\|x_n-q\|^2+\frac{2a_n}{1-a_na}\langle f(q)-q,x_{n+1}-q\rangle$$

$$\le\left[1-\frac{2a_n(1-a)}{1-a_na}\right]\|x_n-q\|^2+\frac{2a_n(1-a)}{1-a_na}\left[\frac{1}{1-a}\langle f(q)-q,x_{n+1}-q\rangle+\frac{a_n}{2(1-a)}M_3\right],\tag{2.12}$$

where M_3 is an appropriate constant such that $M_3\ge\sup_{n\ge1}\left\{\|x_n-q\|^2\right\}$.

Put $j_n=\dfrac{2a_n(1-a)}{1-a_na}$ and $t_n=\dfrac{1}{1-a}\langle f(q)-q,x_{n+1}-q\rangle+\dfrac{a_n}{2(1-a)}M_3$.

Then we have $\|x_{n+1}-q\|^2\le(1-j_n)\|x_n-q\|+j_nt_n.$ (2.13)

It follows from the conditions (i),(ii)and (2.4) that $\lim_{n\to\infty}j_n=0$, $\sum_{n=1}^{\infty}j_n=\infty$, $\limsup_{n\to\infty}t_n\le 0$.

Therefore, applying Lemma1.4 to (2.13), we have $x_n\to q$ as $n\to\infty$. This completes the proof.

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