

Review of Estimation on Vertex Normal Vector and Discrete Curvature of Triangular Mesh

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Abstract. In order to reconstruct the gingival margin curve of the discrete curvature estimation of triangular mesh surfaces, the estimation methods of the vertex normal vector and discrete curvature of the triangular mesh model are summarized and reviewed. By the analysis of the existing method of vertex normal vector and discrete curvature for triangular mesh, estimation calculation of vertex normal vector mainly has three kinds of methods, where the advantages and disadvantages are compared, and estimation methods of discrete curvature for triangular mesh mainly have a local surface fitting, Laplace-Beltrami operator, Moreton and Sequin, and so on. By the review, a more appropriate method is sought for the reconstruction of gingival margin curve.

Introduction

In the dental digital system, the 3D digital dental model is obtained by using the optical measurement technique, and the curved surfaces of the 3D model are expressed in the form of triangular mesh. Curvature is an important feature of a curved surface, and the calculation of curvature has some perfect and mature methods for the surface expressed in parametric form [1]. However, these traditional methods are not suitable to solve the curvature of the surface with discrete representation. The 3D digital dental model is represented by triangular mesh, and the discrete curvature estimation of triangular mesh surface needs to be solved during the reconstruction of the gingival margin curve. The vertex normal vector of the triangular mesh is calculated during the discrete curvature of triangular mesh surface. Therefore, the estimation methods of the vertex normal vector and discrete curvature of the triangular mesh model are analyzed and reviewed in this paper.

Estimation on Vertex Normal Vector of Triangular Mesh

Triangular Mesh Model. Assuming that V_0 is a vertex of triangular mesh and V_i ($i=1, 2, \dots, m$) are connection points of V_0 . n_i ($i=1, 2, \dots, m$) are unit normal vectors of triangular patch, which are shown in Figure 1. The normal vector n_i , which is the normal vector of the triangle patch composed of V_0 , V_i and V_{i+1} , can be expressed as follows:

$$n_i = \frac{(V_i - V_0) \times (V_{i+1} - V_0)}{\|(V_i - V_0) \times (V_{i+1} - V_0)\|} \quad i = 1, 2, \dots, m \quad (1)$$

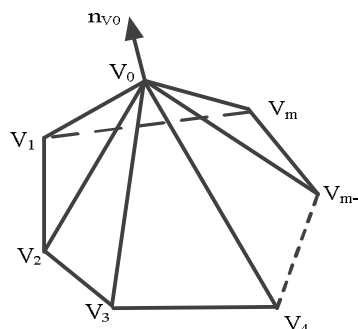


Fig. 1 The normal vector of triangular mesh model

Estimation on Vertex Normal Vector of Triangular Mesh Model. For calculation of the vertex normal vector for the triangle mesh model, the some methods are mainly used as following:

(1) By using the area of each triangle patch for the triangle mesh model as the weight, the unit normal vector of each triangle patch is weighted average, and the vertex normal vector of the triangular mesh model is obtained [2], which can be written as follows:

$$n_{V_0} = \frac{\sum_{i=1}^m A_i n_i}{\left\| \sum_{i=1}^m A_i n_i \right\|} \quad (2)$$

where n_{V_0} is the vertex normal vector of the triangular mesh model, n_i is the normal vector of the triangle patch, and A_i is the area of the triangle patch.

The area of the triangle patch can be expressed as follows:

$$A_i = \frac{(V_i - V_0) \times (V_{i+1} - V_0)}{2} \quad i = 1, 2, \dots, m \quad (3)$$

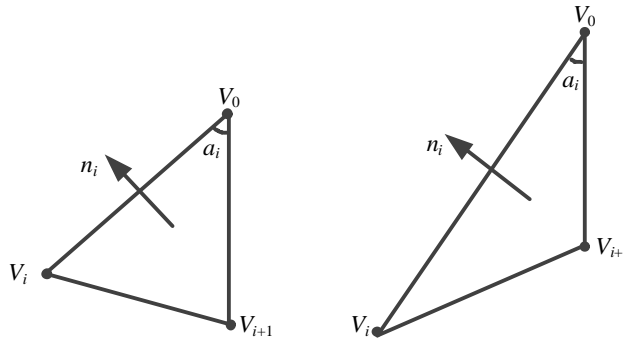


Fig. 2 The same area and normal vector, and the different shape of the triangle patch

The area of the two triangle patches is the same, however, the angle of the triangle patches is different, which may lead to a large difference in the shape (Fig. 1). Therefore, the Eq. (2) does not reflect the influence of the shape of the triangular patch on the vertex normal vector of the triangular mesh model.

(2) In order to reduce the influence of triangle angle on the vertex normal vector of the triangular mesh model, the unit normal vector of each triangle patch is weighted average by using the area and the angle of each triangle patch for the triangle mesh model as the weight [3], and the vertex normal vector of triangular mesh model is obtained, which can be written as follows:

$$n_{V_0} = \frac{\sum_{i=1}^m A_i a_i n_i}{\left\| \sum_{i=1}^m A_i a_i n_i \right\|} \quad (4)$$

where a_i is an angle between the two sides of the vertex V_0 .

(3) Position relation of three vertices of the triangle patch should be shown during the vertex normal vector calculation in the triangular mesh model, and the role of vertices is highlighted. So the unit normal vector of each triangle patch is weighted average by using the distance from the center to the vertex, and the angle of each triangle patch for the triangle mesh model as the weight [4], and the vertex normal vector of the triangular mesh model is obtained, which can be written as follows:

$$n_{V_0} = \frac{\sum_{i=1}^m B_i a_i n_i}{\left\| \sum_{i=1}^m B_i a_i n_i \right\|} \quad (5)$$

where B_i is the distance from the center of the triangle patch to the vertex.

Discrete Curvature Estimation of Triangular Mesh

For the traditional parametric surfaces and implicit surfaces, the calculation of the micro components has a mature theory and method. However, the triangular mesh is a piecewise linear surface, and a classical differential geometry does not provide sufficient theory for the computation of discrete triangular mesh surfaces. At present, the discrete curvature is commonly used calculation methods of a local surface fitting, Laplace-Beltrami operator, Moreton and Sequin, and so on.

Local Surface Fitting Method. A pair of orthogonal vectors u and v is selected on the plane of the normal vector for triangular mesh vertex, and a local coordinate system is established at the vertex of the triangular mesh. For example, the local surface fitting method uses the parabolic surface, which can be written as follows [5]:

$$S(u, v) = au^2 + buv + cv^2 \quad (6)$$

The parabolic surface is fitted by using least square method, and the objective function can be expressed as follows:

$$J_{\min} = \sum_{i=1}^k \left[h_i - (au_i^2 + bu_i v_i + cv_i^2) \right]^2 \quad (7)$$

The maximum and minimum principal curvatures of the vertex of triangular mesh are solved, which can be written as follows:

$$\begin{cases} k_{\max} = a + c + \sqrt{(a - c)^2 + b^2} \\ k_{\min} = a + c - \sqrt{(a - c)^2 + b^2} \end{cases} \quad (8)$$

Laplace-Beltrami Operator. Laplace-Beltrami operator ($\Delta = Hn$) and mean curvature manifold of a surface of the triangular mesh are introduced into the estimation method of discrete Curvature [6]. The estimation method has three forms, namely Taubin method, Mayer method, and Desbrun method.

(1) In Taubin method, Laplace-Beltrami operator is discrete, which is expressed as follows [7]:

$$\Delta = \sum_{j \in N(i)} w_{ij} (p_j - p_i) \quad (9)$$

where p_i and p_j are point on the mesh surface, the weighted factor w_{ij} satisfies $\sum_{j \in N(i)} w_{ij} = 1$.

$$w_{ij} = \frac{t(p_i, p_j)}{\sum_{k \in N(i)} t(p_i, p_k)} \quad (10)$$

where $t(p_i, p_j)$ is the total area of triangles of the edge $p_i p_j$.

(2) In Mayer method, Laplace-Beltrami operator is integrated, which can be written by the Green formula in differential geometry,

$$\int_D \Delta f(x) dx = \int_{\partial D} \partial f(s) ds \quad (11)$$

(3) Desbrun method

Moreton and Sequin Method. By using the Euler theorem of differential geometry, the relationship between the curvature of the surface method, principal curvature, and principal direction is established [6, 8]. The average value of each normal vector of the triangular patches around the vertex v_i is the normal vector of the triangular mesh surface at the vertex v_i , which is expressed as n . The plane through the vertex v_i and the plane perpendicular to the normal vector n is called the tangent plane of the mesh surface, t_j is the unit projection of the vector $v_i v_j$ in the tangent plane of the mesh surface, and the curvature of the curved surface at the vertex v_i along $v_i v_j$ direction can be expressed as follows:

$$k_j = \frac{2n \cdot (v_j - v_i)}{(v_j - v_i) \cdot (v_j - v_i)} \quad (12)$$

Assuming that b_x and b_y are a set of bases on the tangent plane of the mesh surface, t_{jx} and t_{jy} are the coordinates in b_x and b_y , and e_x and e_y are the coordinates of the principal direction in b_x and b_y . According to Euler theorem, the expression is written as follows:

$$k_j = \begin{pmatrix} t_{jx} \\ t_{jy} \end{pmatrix}^T \cdot K \cdot \begin{pmatrix} t_{jx} \\ t_{jy} \end{pmatrix} \quad (13)$$

$$K = \begin{bmatrix} e_x & e_y \\ -e_y & e_x \end{bmatrix} \cdot \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \cdot \begin{bmatrix} e_x & e_y \\ -e_y & e_x \end{bmatrix}^{-1}, \quad i, j = 1, 2, \dots, m, \text{ and the equations are obtained, which can}$$

be written as follows: $Ax = b$, where

$$A = \begin{bmatrix} t_{1x}t_{1x} & t_{1x}t_{1y} & t_{1y}t_{1y} \\ t_{2x}t_{2x} & t_{2x}t_{2y} & t_{2y}t_{2y} \\ \mathbf{M} & \mathbf{M} & \mathbf{M} \\ t_{mx}t_{mx} & t_{mx}t_{my} & t_{my}t_{my} \end{bmatrix}, \quad b = \begin{bmatrix} k_1 \\ k_2 \\ \mathbf{M} \\ k_m \end{bmatrix}, \text{ and } x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e_x^2 k_1 + e_y^2 k_2 \\ e_x e_y (k_1 - k_2) \\ e_x^2 k_2 + e_y^2 k_1 \end{bmatrix}. \text{ So Mean curvature and}$$

Gauss curvature can be obtained, which can be written as follows:

$$K_H = \frac{x_0 + x_2}{2}, \quad KG = 2x_0x_2 - \frac{x_1^2}{2} \quad (14)$$

Conclusions

In this paper, a conclusion is drawn that estimation calculation of vertex normal vector mainly has three kinds of methods, and estimation methods of discrete curvature for triangular mesh mainly have the local surface fitting, Laplace-Beltrami operator, Moreton and Sequin, and so on. The advantages and disadvantages of those methods are summarized and compared, and a more appropriate method is sought for the reconstruction of gingival margin curve by the analysis of the existing method of vertex normal vector and the discrete curvature for triangular mesh.

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