Mechanical Proving for ERDÖS-SZEKERES Problem

[1]Meijing Shan

Institute of Information science and Technology,

East China University of Political Science and Law, Shanghai, China. 201620

shanmeijing@ecupl.edu.cn

Abstract： The Erdös-Szekeres problem was an open unsolved problem in computational geometry and related fields from 1935. Many results about it have been shown. The main concern of this paper is not only show how to prove this problem with automated deduction methods and tools but also contribute to the significance of automated theorem proving in mathematics using advanced computing technology. The present case is engaged in contributing to prove or disprove this conjecture and then solve this problem. The key advantage of our method is to utilize the mechanical proving instead of the traditional proof and this method could improve the arithmetic efficiency.

Introduction

The following famous problem has attracted more and more attention of many mathematicians [3, 6, 12, 16] due to its beauty and elementary character. Finding the exact value of $N(n)$ turns out to be a very challenging problem. The problem is very easy to explain and understand.

The Erdös-Szekeres Problem 1.1 [4, 15]**.** For any integer $n \geq 3$, determine the smallest positive integer $N(n)$ such that any set of at least $N(n)$ points in *generalposition* in the plane contains n points that are the vertices of a convex n-gon.

A set of points in the plane is said to be in the *general position* if it contains no three points on a line. This problem was also called *Happy Ending Problem* by Erdös, because two investigators Esther Klein and George Szekeres who first worked on the problem became engaged and subsequently married[8, 17].

The interest of Erdös and Szekeres in this problem was initiated by Esther Klein(later Mrs. Szekeres), who observed that from 5 points of the plane of which no three lie on the same straight line it is always possible to select 4 points determining convex quadrilateral. There are three distinct types of five points in the plane, as shown in Figure 1.
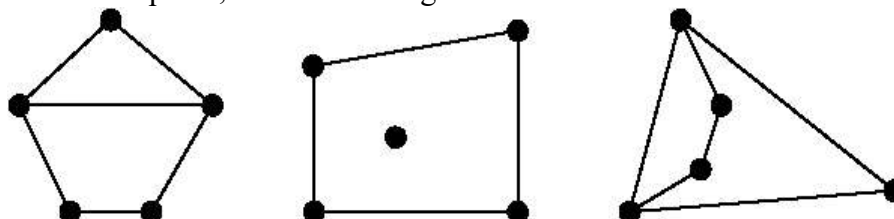
Figure 1. Three cases for 5 points.

In any case of the Figure 1, one can find at least one convex quadrilateral determined by the points. Klein [4] suggested the following more general problem: Can we find for a given n a number $N(n)$ such that from any set containing at least Npoints it is possible to select n points forming a convex polygon?

As observed by Erdös and Szederes [4], there are two particular questions:

(1) Does the number N corresponding to n exist?

(2) If so, how is the least $N(n)$ determined as a function of n?

They proved the existence of the number $N(n)$ by two different methods. The first one is a combinatorial theorem of Ramsey. The foundation of the second one isbased on some geometrical and combinatorial considerations. And then they formulated the following conjecture.

Conjecture 1.1 $N(n) = 2^{n-2} + 1$ for all $n \geq 3$.

Despite its elementary characters and the efforts of many researchers, the Erdös Szekeres problem is solved for the value $n = 3$, 4 and 5 only. The case $n = 3$ is trivial, and $n = 4$ is due to Klein. The equality $N(5) = 9$ was proved by E. Makai while the published proof by Kalbfleisch [11] and then Bonnice [2] and Lovasz [13]independently published the much simpler proofs. The bottle neck of this problem now is when $n > 5$, how to prove or disprove the conjecture.

About this problem, the best currently known bounds are

$$2^{n-2} + 1 \leq N(n) \leq \binom{2n-5}{n-2} + 2, \tag{1}$$

Where $\binom{n}{k}$ is a binomial coefficient. The lower bound was obtained by Erdös andSzekeres [4] and the upper bound is due to Töth and Valtr [18]. The lower bound issupposed to be sharp, according to conjecture 1.1.

In this paper we use the automated deduction method and tools [1, 19-21] to establish a mechanical method for proving problem instead of the manual proof. We hope the method might give a rise to substantially promote this unsolved problem.The rest content of the paper is indicated by section headings as follows: Section 2 preliminaries, Section 3 main results, Section 4 conclusion and remarks.

Preliminaries

In this Section, we present some algorithms that would help us develop our method in next section.

Algorithm 2.1. Modified Cylindrical Algebraic Decomposition (CAD). Due to the problem statement, the proof should consider all kinds of points' positions on the plane.

The Cylindrical Algebraic Decomposition [5, 14] of $R\,n$ adapted to a set ofpolynomials which is a partition of $R\,d$ is cells (simple connected subsets of $R\,d$)such that each input polynomial has a constant sign on each cell. Basically, thealgorithm computes recursively at least one point in each cell (so that one can testthe cells that verify a fixed sign condition). The sample points Pi got by originalCAD are always more than one on each cell. We modify the procedure to have asample point on each cell of the final cell decomposition, by the rule that Pi has a constant sign on each cell. We elaborate the main idea under lying our method by showing how our main algorithm evolved from the original one.

Here, we describe it as follows.

Algorithm. MCAD

Input: A set F of polynomials

Output: A F-sign-invariant CAD of $R\,n$

Step 1. Projection. Compute the projection polynomials which using exclusively operations, and receive some $(n-1)$ -variate polynomials.

Step 2. Recur. Apply the algorithm recursively to compute a CAD of $R\,n-1$which $Q(F)$is sign-invariant.

Step 3. Lifting. Lift the $Q(F)$-sign-invariant CAD of $R\,n-1$ up to aF-sign-invariant CAD of using the auxiliary polynomial $\Pi(F)$ of degree no largerthan $d(F)$ (d is the maximum degree of any polynomial in F).

Step 4. Choice. Utilize the strategy that $\{Pi\}$ has constant sign on each cell tochoose one sample point on each cell.

Algorithm 2.2 (Graham Scan Algorithm) [9, 10, 22]

We present one of the simplest algorithms used to find the convex hull fromsome points. Some basic definitions are provided in the field of Computational Geometry. This algorithm works in three phases:

Input: A set S of points

Output: The convex hull of S.

Step 1. Find an extreme point. The algorithm starts by picking a point in S known to be a vertex of the convex hull. This point is chosen to be with smallest y coordinate and guaranteed to be on the hull. If there are some points with the same smallest y coordinate, we will choose the point with largest x coordinate in them. In other words, we select the right most lowest point as the extreme point.

Step 2. Sort the points. Having selected the base point which is called $P0$, then the algorithm sorts the other points P in S by the increasing counter-clockwise (ccw) angle the line segment $P0P$ makes with the x-axis. If there is a tie and two points have the same angle, discard the one that is closest to $P0$.

Step 3. Construct the convex hull. Build the hull by marching around the star shaped polygon, adding edges when we make a left turn, and back-tracking when we make a right turn. We end up with a star-shaped polygon, see Figure 3 (one in whichone special point, in this case the pivot, can "see" the whole polygon). Considering efficiency in Step 2, it is important to note that the comparison of sorting between two points $P2$ and $P3$ can be made without actually computing their angles. In fact, computing angles would use slow in accurate trigonometry functions, and doing these computations would be a bad mistake. Instead, one just observes that $P2$ would make a greater angle than $P1$ if (and only if) $P2$ lies on theleft side of the directed line segment $P0P1$, see Figure 2.

We make full use of this algorithm to judge whether the polygon received in every recursive step is a convex polygon or not. It is a decision method in our algorithm.
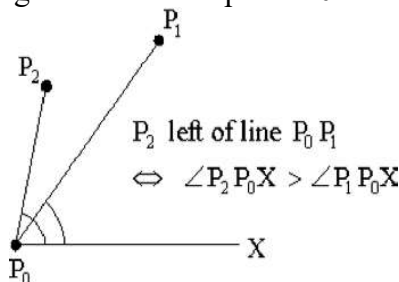
To state the algorithm clearly, we will describe it in a style of pseudo-code.

Algorithm. Graham Scan Algorithm

Input: A set S of points in the plane

Output: A list containing the vertex of the convex hull

Select the right most lowest point $P0$ in S

Figure 2. Sort the points.

Figure 3. Graham Scan

Sort S angularly about $P0$ as a center.

For ties, discard the closer points.

Let **S** be the sorted array of points.

Push $S[1] = P0$ and $P1$ onto a stack Ω.

Let $P1$ = the top point on Ω

Let $P2$ = the second top point on Ω

    while $S[k]$ P do

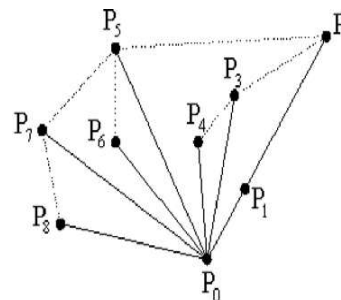   if ($S[k-1]$ is strictly left of the line $S[k]$ to $S[k+1]$), then

            Push $S[k-1]$ onto Ω

      else

            Pop the top point $S[k]$ off the stack Ω.

        fi;

      od;

Main Results

In contrast with the traditional proof, this method presented following can show the convex polygons received in every step.

The main idea of our algorithm: First, give randomly four points in general position in the plane, and then we use polynomials of points' coordinates to represent the lines. We extend the 4-element set to some5-element sets. We do this by establishing the corresponding Modified Cylindrical Algebraic Decomposition (MCAD) and design an interactive program which allow the user to choose among the candidate one sample point in each cell. We use Graham Scan algorithm to determine whether or not there is a convex 5-gon (convexpentagon) in every set received. If some of the received 5-element sets have no convex 5-gon, then extend them to the 6-element sets by the strategy mentionedabove. Simultaneously, check whether or not each of the received 6-element sets has any convex hull at least with any convex 5-gon in the polygon. We implement theprogram repeatedly until can find a convex n-gon ($n \geq 5$) in any set. We trace the processing of the extension and decision, and then draw a conclusion that $N(5) = 9$.

To prove this approach, we write the following algorithm named "conv5". Based on this algorithm can generate short and readable mechanical proving for the Erdös-Szekeres conjecture, including the case of $n = 3, 4, 5$. It consists of two main algorithms—Modified Cylindrical Algebraic Decomposition algorithm and Graham Scan algorithm and some sub-algorithms such as collinear, pol, sam, ponlist, min0,isleft, ord, conhull, point5, convex, G5,G6,Pmn.

Algorithm Conv5

Input: four points in the general position in the plane.
Output: Any polygon with at least 9 points in general position in the planecontains a convex 5-gon.
Step 1 [collinear]**.** Write the line polynomials with the given four points (basepoints).
Step 2 [pol, sam, ponlist]**.** Illustrate how we utilize the CAD to find some
sample points in the cell which built by the lines, and then with the base n points get$n$ +1 -element sets.
Step 3 [min0, isleft, conhull, point5, convex, G5]**.** Decide whether or not thereis a convex hull or a convex n-gon ($n \geq 5$) in every set; if it is true, then stop; elsegoto Step 4.
Step 4 [G6, Pmn]**.** Deal with the sets which have no convex n-gon ($n \geq 5$).
Recursively implement Step 2 and Step 3 process, until at least there is a convex 5-gon in any set.
End Conv5.
The key techniques of the algorithm are listed as follows:
1. To reduce the complexity of the computation and increase the efficiency,when we check whether or not there is a convex 5-gon in the given points set,we utilize the strategy as follows:if there is a convex hull at least with 5 points in the points set thenpop this points setelif there is a convex 5-gon thenpop this points setelse "there is no convex 5-gon in this points set" go to next step
2. Each convex n-gon ($n \geq 5$) contains a convex 5-gon
3. If there is no convex 4-gon, then there should be no convex 5-gon.

Conclusion and Remarks

By the Maple procedure we have implemented the mechanical method for the conjecture in certain cases. Through observing the whole computational process, we obtain a certain answer that any set with at least 9 points in general position in the plane contains a convex 5-gon. This method can be generalized in an obvious way to arbitrary base points in the plane.

For the mechanical method proposed here, on one hand it provided one of the promising direction for proving or disproving the Conjecture 1.1 (when $n \geq 6$), even for handling with some unsolved problems in computational geometry. On the other hand, it gave one especially useful application of computer algebra andautomated deduction. For further investigations, now we consider about the following problems:
1. Does any set of at least 17 points in general position in the plane contains 6 points which are the vertices of a convex hexagon? Can we give the proof about $N(6)$ existence and prove or disprove

the corresponding conclusion by mechanical proving? Now the best known conclusion about this is $N(6) \geq 27$, if it exists.

2. Erdös posed a similar problem on empty convex polygons. Whether or not we can give the automated proof to this problem?

References

[1] M. de Berg, M. van Kreveld, M. Overmars and O. Schwarzkopf, omputational GeometryAlgorithm and Applications, (2nd ed.), Spring-Verlag, Berlin, Heideberg, New York, 1997.

[2] W. E. Bonnice, On convex polygons determined by a finie planar set, Amer. Math. Monthly.

[3] F. R. K. Chung and R. L. Graham, Forced Convex n-gons in the Plane, Discr. Comput. Geom. 19(1998), 367-371.

[4] P. Erdös and G. Szekeres, A combinatiorial problem in geometry, Comositio Mathematica 2(1935), 463-470.

[5] E. Collins George, Quantifier elimination for the elementary theory of real closed fields by cylindrical algebraic decomposition Lecture Notes In Computer Sciencevol. 33, Springer-Verlag, Berlin, pp. 134-183.

[6] Kráolyi Gyula, An Erdös-Szekeres type problem in the plane.

[7] X. R. Hou and Z. B. Zeng, An efficient Algorithm for Finding Sample Points of Algebraic Decomposition of Plane, Computer Application, 1997, (in Chinese).

[8] P. Hofiman, The Man Who loved Only Numbers Hyperion, New York, 1998.

[9] http://www.cs.princeton.edu/ah/alganim/version0/Graham.html.

[10] http://cgm.cs.mcgill.ca/ beezer/cs507/main.html.

[11] J. D. Kalbfleisch, J. G. Kalbfleisch and R. G. Stanton, A combinatorial problem on convexregions, Proc. Louisiana Conf. Combinatorics, Graph Theory and Computing, Louisianna StateUniv., Baton Touge, La, Congr. Numer. 1 (1970), 180-188.

[12] D. Kleitman and L. Pachter, Finding convex sets among points in the plane, Discr. Comput.Geom. 19, (1998), 405-410.

[13] L. Lovasz, Combinatorial Problem and Exercises North-Holland, msterdam, 1979.

[14] Bhubaneswar Mishra, Algorithmic Algebra, Springer-Verlag, 2001.

[15] W. Morris and V. Soltan, The Erdös-Szekeres Problem on Points in Convex

Position- A, Survey Bulletin of the American Mathematical Society, vol. 37.

[16] L. Graham Ronald and Yao Frances, Finding the Convex Hull of a Simple

Polygon Report No. STAN-CS-81-887, 1998.

[17] B. Schechter, My Brain is Open Simon Schuster, New York, 1998.

[18] G. Tóth and P. Valtr, Note on the Erdös-Szekeres theorem, Discr. Comput. Geom. 19 (1998), 457-459.

[19] W. T. Wu, Basic Principles of Mechanical Theorem Proving in Geometries Science Press,Beijing, 1984, (Part on elementary geometries, in Chinese).

[20] L. Yang and B. C. Xia, Automated Deduction in Real Geometry, Geometric Computation, WorldScientific, 2004.

[21] L. Yang and Z. Z. Jing and X. R. Hou, Nonlinear Algebraic Equation System and AutomatedTheorem Proving, Shanghai Scientific and Technological Education Publishing House, Shanghai,1996, (in Chinese).

[22] P. D. Zhou, Computational Geometry Design and Analysis in Chinese, TsingHua UniversityPress, 2005.