

L1-induced Performance Analysis for Positive Systems with Interval Uncertainties

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Abstract. The L_1 performance analysis is studied for continuous-time interval positive systems. In this paper, necessary and sufficient conditions for stability and L_1 performance are established for positive linear systems through linear inequalities.

Introduction

In real world, positive systems exist in many branches of technology and science such as industrial engineering [1], ecology [2]. For non-negative initial conditions and inputs, the variables of positive systems always take non-negative values. Positive systems are special systems and they are defined on cones instead of linear spaces. Therefore, many well-established approaches to system analysis and synthesis cannot give satisfactory results, which motivates the research on positive systems.

Recently, many research results on positive systems have been reported in the literature [4–11] after the development of a system-theoretic approach to positive systems in [3]. For example, controllability and reachability for positive systems have been studied in [14] and [15]. The positive state-space representation for a given transfer function has been proposed in [12]. For non-negative and time-delay compartmental dynamic systems, stability has been thoroughly studied in [18–19]. Research on 2-D positive systems has appeared in [7]. Conditions for positive realizability in terms of convex analysis have been presented in [13]. The design of state-feedback controllers guaranteeing the closed-loop system to be asymptotically stable has been studied by the linear programming approach in [16]. Moreover, the problem of controller design for positive systems has been investigated by the linear matrix inequality (LMI) approach in [17].

In recent years, many novel results through linear Lyapunov functions have appeared [17]. As is well known, a lot of existing results about positive systems are derived via quadratic Lyapunov functions and many effective methods are available. In comparison with the quadratic Lyapunov function based approach, the novel method is more simple and amenable to computation and analysis. The reason we use the novel approach is that the variables of positive systems are non-negative and therefore a linear Lyapunov function becomes valid.

As is well known, it is inevitable that uncertainties enter the system parameters due to the fact that practical systems are usually influenced by variations, disturbances or environmental perturbations. For interval positive systems, the L1-induced performance analysis deserves to be studied and is challenging, due to the complexity caused by parameter uncertainties.

In this paper, the L1-induced performance analysis problem is considered for interval positive systems. In detail, for a given positive system, the objective is to propose an L1-induced performance index. In Section 2, the L1-induced performance is first established and some preliminary results are proposed. In Section 3, a performance characterization is derived, under which the positive system is robustly stable with the L1-induced performance. We present the conclusion of this paper in Section 4.

Preliminaries

In this section, many results are introduced for positive systems. First, we present the following notations.

\mathbb{R} denotes the set of real numbers; $A \geq B$ (respectively, $A > B$) means that $A - B \geq 0$ (respectively, $A - B > 0$). For $A \in \mathbb{R}^{m \times n}$, a_{ij} is the element located at the i th row and the j th column. The notation $A \geq B$ (respectively, $A > B$) means that the matrix $A - B \geq 0$ (respectively, $A - B > 0$). The L_1 -norm of x is defined as $\|x\|_{L_1} = \int \|x(t)\|_1 dt$. We denote $\mathbf{1} = [1, 1, \dots, 1]^T$. The induced 1-norm of a matrix $Q = [q_{ij}] \in \mathbb{R}^{m \times n}$ is denoted by $\|Q\|_1 = \max(\sum |q_{ij}|)$. $A \in \mathbb{R}^{n \times n}$ is Metzler, if all its off-diagonal elements are positive. That is, $\forall(i, j), i \neq j, a_{ij} \geq 0$. $A \geq 0$ (respectively, $A > 0$) means that for all i and j , $a_{ij} \geq 0$ (respectively, $a_{ij} > 0$). \mathbb{R}^n denotes the set of n -column real vectors. The induced 1-norm of a matrix $Q = [q_{ij}] \in \mathbb{R}^{m \times n}$ is denoted by $\|Q\|_1 = \max(\sum |q_{ij}|)$. For matrices $A, \underline{A}, A \in \mathbb{R}^{n \times m}$, the notation $A \in [\underline{A}, A]$ means that $\underline{A} \leq A \leq A$. The symbol $\rho(A)$ denotes the spectral radius of matrix A , that is, $\rho(A) = \max\{|\lambda_i(A)|, i = 1, 2, \dots, n\}$. The symbol $\text{col}_i(A)$ denotes the i th column of matrix A . The L_1 -norm of x is defined as $\|x\|_{L_1} = \int \|x(t)\|_1 dt$. We denote $\mathbf{1} = [1, 1, \dots, 1]^T$. $A \geq 0$ (respectively, $A > 0$) means that for all i and j , $a_{ij} \geq 0$ (respectively, $a_{ij} > 0$). The superscript “ T ” denotes matrix transpose. $\mathbb{R}^{n \times m}$ is the set of all real matrices of dimension $n \times m$. $\|\cdot\|$ represents the Euclidean norm for vectors. The 1-norm of a vector $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ is defined as $\|x(t)\|_1 = \sum |x_i(t)|$. \mathbb{N} is the set of natural numbers. The spectral abscissa of matrix A is denoted by $\alpha(A) = \max\{\text{Re}\lambda_i(A), i = 1, 2, \dots, n\}$.

Consider a system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + B_w w(t), \\ z(t) &= C_z x(t) + Du(t) + D_{zw} w(t), \\ y(t) &= Cx(t),\end{aligned}\tag{1}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^n, w(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^r, z(t) \in \mathbb{R}^q$ are the state, control input, disturbance input, measurement and controlled output, respectively.

The following definition is introduced which will be used in the paper.

Definition 1 The continuous-time linear system (1) is said to be positive if for all initial state $x(0) \geq 0$ and disturbance input $w(t) \geq 0$, we have $x(t) \geq 0, z(t) \geq 0$ and $y(t) \geq 0$ for $t > 0$.

Next, some useful results are introduced which will be used later in this paper.

Proposition 1 For a positive system (1), we say it is stable if and only if there exists a nonnegative vector p satisfying the following inequality:

$$p^T A \leq 0.$$

Lemma 1 The continuous-time system (1) is positive if and only if A is M-matrix, B_w, C_z, D_{zw} , and C are all nonnegative.

Then, the definition of L_1 -induced norm is proposed in the following.

For a stable positive system in (1) with zero initial conditions, it has L_1 -induced performance at the level γ if,

$$\|z\|_{L_1} < \gamma \|w\|_{L_1}\tag{2}$$

under zero initial conditions.

Main Results

The following theorem is proposed and it serves as a characterization on the stability and performance of system (1).

Theorem 1. For a positive system in (1), it is stable with the performance in (2) satisfied under initial conditions if and only if we can find a nonnegative vector p satisfying

$$\begin{bmatrix} p^T & 1^T \end{bmatrix} \begin{bmatrix} A & B_w \\ C_z & D_{zw} - \gamma I \end{bmatrix} \ll 0. \quad (3)$$

Next, the following system is considered:

$$\begin{aligned} \dot{x}(t) &= A_I x(t) + B_{Iu}(t) + B_{Iw} w(t), \\ z(t) &= C_{Iz} x(t) + D_{Iz} u(t) + D_{Izw} w(t), \\ y(t) &= C_I x(t), \end{aligned} \quad (4)$$

with

$$A_I \in [\underline{A}_I, \overline{A}_I], B_{Iw} \in [\underline{B}_{Iw}, \overline{B}_{Iw}], C_{Iz} \in [\underline{C}_{Iz}, \overline{C}_{Iz}], D_{Izw} \in [\underline{D}_{Izw}, \overline{D}_{Izw}], C_I \in [\underline{C}_I, \overline{C}_I] \quad (5)$$

Next, one definition about interval positive system is presented.

Definition 2. The interval positive system in (4) is robustly stable if it is stable over the whole uncertainty domain in (6).

Then, the following theorem is established as the performance characterization result for interval positive system in (4).

Theorem 2. The interval positive system in (4) is robustly stable and satisfies the L_1 -induced performance for any $A_I \in [\underline{A}_I, \overline{A}_I], B_{Iw} \in [\underline{B}_{Iw}, \overline{B}_{Iw}], C_{Iz} \in [\underline{C}_{Iz}, \overline{C}_{Iz}], D_{Izw} \in [\underline{D}_{Izw}, \overline{D}_{Izw}], C_I \in [\underline{C}_I, \overline{C}_I]$ if and only if there exists a nonnegative vector p satisfying the following inequalities:

$$\begin{bmatrix} p^T & 1^T \end{bmatrix} \begin{bmatrix} \overline{A}_I & \overline{B}_{Iw} \\ \overline{C}_{Iz} & \overline{D}_{Izw} - \gamma I \end{bmatrix} \ll 0 \quad (6)$$

Proof:

For any $A_I \in [\underline{A}_I, \overline{A}_I], B_{Iw} \in [\underline{B}_{Iw}, \overline{B}_{Iw}], C_{Iz} \in [\underline{C}_{Iz}, \overline{C}_{Iz}], D_{Izw} \in [\underline{D}_{Izw}, \overline{D}_{Izw}], C_I \in [\underline{C}_I, \overline{C}_I]$, we have

$$\begin{aligned} \begin{bmatrix} p^T & 1^T \end{bmatrix} \begin{bmatrix} A \\ C_z \end{bmatrix} &\leq \begin{bmatrix} p^T & 1^T \end{bmatrix} \begin{bmatrix} \overline{A}_I \\ \overline{C}_{Iz} \end{bmatrix} \ll 0, \\ \begin{bmatrix} p^T & 1^T \end{bmatrix} \begin{bmatrix} B_w \\ D_{zw} - \gamma I \end{bmatrix} &\leq \begin{bmatrix} p^T & 1^T \end{bmatrix} \begin{bmatrix} \overline{B}_{Iw} \\ \overline{D}_{Izw} - \gamma I \end{bmatrix}, \end{aligned}$$

Therefore, it is noted that the interval system in (4) is robustly stable and satisfies the L_1 -induced performance under initial conditions. The sufficiency is proved.

Next, we assume that the interval system in (4) is robustly stable and satisfies the L_1 -induced performance under initial conditions. The following inequalities hold.

$$\begin{bmatrix} p^T & 1^T \end{bmatrix} \begin{bmatrix} A & B_w \\ C_z & D_{zw} - \gamma I \end{bmatrix} \ll 0$$

Then, the inequality (6) naturally holds.

Summary

In this paper, the L_1 -induced performance analysis problem has been studied for interval positive systems. We have established a novel performance characterization to guarantee that the interval positive system is robust stable and satisfies the L_1 -induced performance.

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