Study on the Unified Mathematical Model for All Types of Cam Mechanisms

Wei He[1,2,a], Xueming He[2,b*], Feng Xu[1,c], Jiujun Zhen[1,d], Rong Zhang[3,e]

[1]Mechanical Engineering Department of Nanjing Institute of Industry Technology, Nanjing 210023, China

[2]Jiangsu Key Laboratory of Advanced Food Manufacturing Equipment and Technology, Jiangnan University, Wuxi, 214122, China

[3]School of Science, Jiangnan University, Wuxi, 214122, China

[a]hewei1957@163.com, [b]hxueming@163.com, [c]xuf@niit.edu.cn, [d] zhengjj@niit.edu.cn, [e]zhr89@163.com

Abstract. This paper presents the unified mathematical model well applicable for the analysis and design of the planar and spatial cam mechanisms of all categories in terms of the differential geometry and meshing principle, exploring the interrelation of the various cam mechanisms. The general-purpose CAD/CAM software package for cam mechanisms is made possible by utilizing the unified mathematical approach developed in this paper. The normalization of the initial parameters for the CAD of certain representative cam mechanisms is also examined.

Notations

C constant vector

C_1 transformation vector of translation from S_f to S_1

C_2 transformation vector of translation from S_2 to S_c

$E^{j\theta_2}, E^{-k\theta_1}$ rotation matrices for the rotation transformations of S2 about the j axis and S_f about the k axis

h_1 follower displacement

h_2 cam displacement

h_{1x}, h_{1y}, h_{1z} components of any follower displacement h_1 in space in   the direction x_f, y_f and z_f

J_1, J_2 2nd and the 3rd component of the rotation matrix

j axis x_2 or y_2 or z_2

k axis x_1 or y_1 or z_1

P contact point

R_f vector notation of Σ_1 in S_f

R_1 vector notation of Σ_1 in S_1

R_2 vector notation of Σ_1 in S_2

R_c vector notation of Σ_1 in S_c

r_f follower rotation radius

$r_0=r_0(\delta_f)$ element function of a specific rotating curved surface

$S_1 (o_1, x_1, y_1, z_1)$ fixed system on the initial position of follower.

$S_f (o_f, x_f, y_f, z_f)$ motive system on the follower with the initial position lying on S_1.

$S_2 (o_2, x_2, y_2, z_2)$ fixed system on the initial position of cam, whose direction is consistent with S_1.

$S_c (o_c, x_c, y_c, z_c)$ motive system on cam, whose initial position lies on S_2.

t time

V_1, V_2 translatory velocities of follower and cam.

Σ_1 mating surface of follower

Σ_2 cam curved surface (curve)

φ, δ_f curved vein coordinates of the roller curved surface.

β constant angle, $\beta = 0°$ or $\beta = 90°$

δ_{fx}, δ_{fy}, δ_{fz} curve vein coordinates of Σ_1

θ_1, θ_2 angular displacements of follower and cam

ϕ contact function, $\phi(\delta_f, \varphi, t) = 0$

ω_1, ω_2 angular velocities of follower and cam

ρ_1, ρ_2, $-\rho_3$ components of O_1O_2 in the direction of x_1, y_1 and z_1, respecti

Introduction

The traditional design and manufacture of cam mechanisms have not met the requirements of industry. Although the CAD/CAM technology of cams develops rapidly, nevertheless a number of CAD software for cam mechanisms find their limited application to some extent due to a great diversity and variation of cam mechanisms. This paper makes an approach to the development of the unified mathematical model, the design and the analysis of cam mechanisms.

Formulation of unified mathematical model

(a)Universal geometric model of cam mechanisms

The implement of motion specifications of cam mechanisms is dependent on the cam profile or the shape of working curved surface and the arrangement of mechanism configuration. For this reason, it is necessary to formulate the mathematical model of the cam profile or the curved surface while we proceed to the precise design of cam mechanisms by using the CAD technique. First of all, a universal geometric model for cam mechanism is constructed. As shown in Figure 1, the geometric model is represented by the kinematics schematic diagram of the spatial cam mechanism that is most general. This model can be converted to the planar or spatial cam mechanisms with the diverse configuration and the forms of motion after the different initial parameters are chosen. The four different Cartesian coordinate systems are established in Figure 1 to derive the accurate mathematical formula of cam profile or the working curved surface of cam:

S_1 (o_1, x1, y1, z1)— a fixed system on the initial position of follower.

S_f (o_f, x_f, y_f, z_f)— a motive system on the follower with the initial position lying on S_1.

S_2 (o_2, x_2, y_2, z_2)— a fixed system on the initial position of cam ,whose direction is consistent with S_1.

S_c (o_c, x_c, yc, zc)— a motive system on cam, whose initial position lies on S_2.

On the basis of the envelope theory of differential geometry, the coordinate transformation can be made by utilizing the rotation matrix or the translation of coordinates in such a manner that the vector notation of the mating surface Σ_1 of follower in S_f relative to the contact point P is transformed to that in S_c relative to the cam working curved face as the roller curved faces of the family as the envelope surface. Then the contact equation can be derived by dint of meshing principle (contact condition) again, resulting in the unified mathematic expression for the cam-curved surface (curve) Σ_2, that is, $\Sigma_1 \xrightarrow{l(p)} \Sigma_2$. The coordinate transformation of vector is

$$\boldsymbol{R}_f^{(P)} \rightarrow \boldsymbol{R}_1^{(P)} \rightarrow \boldsymbol{R}_2^{(P)} \rightarrow \boldsymbol{R}_c^{(P)}$$

where these vectors above can be simply rewritten as: $\boldsymbol{R}_f, \boldsymbol{R}_1, \boldsymbol{R}_2,$ *and Rc.*
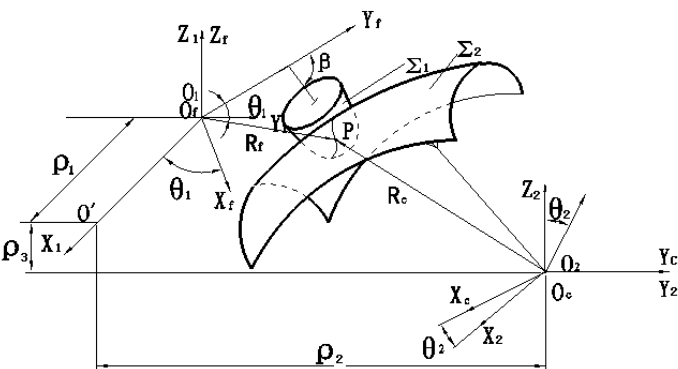
Fig.1 The universal geometric model of cam mechanism

(b)Vector notation $\boldsymbol{R}_f$ of Σ_1 in S_f

There are many forms of engagement or contact between a cam curved surface and a follower in direct contact, such as the roller, the flat-faced, the knife-edge, etc. Among them the roller follower is most complicated, consisting of the cylindrical roller, the conic roller, the spherical roller and the other kinds of rollers with rotating curved face, etc. However, no matter what classes of roller, Σ_1 is usually a given simple curved face .The mathematical expression of vector $\boldsymbol{R}_f$ in S_1 is generally simple.

Let Σ_1 be any specific rotation curved surface with the roller follower. As the rotating of roller about its axis has no effect upon the forms of engagement, the roller can be supposed to be fixed on the follower as shown in Figure 2. In this case, the vector $\boldsymbol{R}_f$ for Σ_1 is as follows

$$\boldsymbol{R}_f = \begin{Bmatrix} x_f \\ y_f \\ z_f \end{Bmatrix} = \begin{Bmatrix} r_0 \sin\varphi \\ (\delta_f + b)\cos\beta + r_0\cos\varphi + r_f \\ -(\delta_f + b)\sin\beta \end{Bmatrix} \tag{1}$$

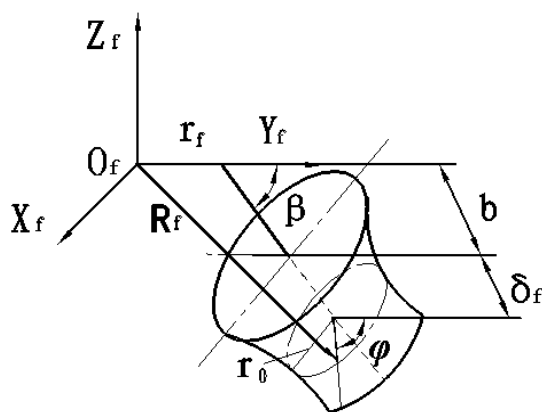Fig. 2 Vector $\boldsymbol{R}_f$ for roller curved surface Σ_1

where $r_0=r_0(\delta_f)$ is the element function of a specific rotating curved surface , φ and δ_f represent the curved vein coordinates of the roller curved surface.

Since the follower is in the translational motion for the flat-faced follower, β is either 0 or 90°.As a result, when $\beta=0°$,

$$\boldsymbol{R}_f = \begin{Bmatrix} x_f \\ y_f \\ z_f \end{Bmatrix} = \begin{Bmatrix} \delta_{fx} \\ r_f + b \\ \delta_{fz} \end{Bmatrix} \tag{2}$$

and when $\beta=90°$,

$$\boldsymbol{R}_f = \begin{Bmatrix} x_f \\ y_f \\ z_f \end{Bmatrix} = \begin{Bmatrix} \delta_{fx} \\ \delta_{fy} \\ b \end{Bmatrix} \tag{2'}$$

where δ_{fx} and δ_{fz} or δ_{fx} and δ_{fy} are the curve vein coordinates of Σ_1 For the knife-edge follower, $r_0 = \delta_f = 0$, therefore, $\boldsymbol{R}_f$ is a constant vector:

$$\boldsymbol{R}_f = \begin{Bmatrix} x_f \\ y_f \\ z_f \end{Bmatrix} = \begin{Bmatrix} 0 \\ b\cos\beta + r_f \\ b\sin\beta \end{Bmatrix} \tag{3}$$

The $\boldsymbol{R}_f$ above are also suitable for the planar cam mechanisms. In this instance, assuming $z_f=0$ and $\beta=0$, $\boldsymbol{R}_f$ can be reduced to the two-dimensional vectors. It must be noted that although only the roller follower was previously illustrated for a clear visualization's sake, the model shown in Figure 1 is likewise suited for the other forms of contact.

(c) Transformation of coordinates

In order to derive the mathematical formula of $\boldsymbol{R}_c$, the transformation of the four coordinate systems from S_f to S_c will be necessarily made. The transformation of the coordinate from S_o to S_1 depends on the motion form of follower in addition that the transformation from S_1 to S_2 needs to be added a constant vector $\boldsymbol{C}$. On the other hand, the coordinate transformation of S_2 to S_c depends on the motion form of cam. In spite of the fact that there are a wide variety of forms in the motion of cam mechanisms, the unified mathematical model can be obtained from the following several vectors $\boldsymbol{R}_c$.

(1) If the cam rotates about the j axis (axis x_2 or y_2 or z_2) and the follower rotates about the k axis (axis x_1 or y_1 or z_1), then

$$\boldsymbol{R}_c = E^{j\theta_2}(E^{-k\theta_1}\boldsymbol{R}_f - \boldsymbol{C}) \tag{4a}$$

where $E^{j\theta_2}, E^{-k\theta_1}$ are the rotation matrices for the rotation transformations of S_2 about the j axis and S_f about the k axis, θ_1 and θ_2 are the angular displacements of follower and cam, respectively.

(2) If the cam rotates about the j axis and the follower translates (in-line), then

$$\boldsymbol{R}_c = E^{j\theta_2}(\boldsymbol{R}_f + \boldsymbol{C}_1 - \boldsymbol{C}) \tag{4b}$$

where $\boldsymbol{C}_1 = \boldsymbol{C}_1(h_1)$ is a transformation vector of translation from S_f to S_1, h_1 is the follower displacement.

(3) If the cam translates and the follower rotates about the k axis, then

$$\boldsymbol{R}_c = (E^{-k\theta_1}\boldsymbol{R}_f - \boldsymbol{C}) - \boldsymbol{C}_2 \tag{4c}$$

where $-\boldsymbol{C}_2 = -\boldsymbol{C}_2(h_2)$ is a transformation vector of translation from S_2 to S_c, h_2 is the cam displacement.

(4) If the cam translates and the follower translates (in-line), then

$$\boldsymbol{R}_c = [(\boldsymbol{R}_f + \boldsymbol{C}_1) - \boldsymbol{C}] - \boldsymbol{C}_2 \tag{4d}$$

These classes discussed above are the most common forms in the motion of cam mechanisms. Especially, when the cam contains more complex motions, i.e. both the rotary and the translating one, the additional cases results:

(5) If the cam rotates and translates, and the follower rotates or oscillates about the k axis, then

$$\boldsymbol{R}_c = E^{j\theta_2}(E^{-k\theta_1}\boldsymbol{R}_f - \boldsymbol{C}) - \boldsymbol{C}_2 \tag{4e}$$

(6) If the cam rotates and translates, and the follower translates, then

$$\boldsymbol{R}_c = E^{j\theta_2}[(\boldsymbol{R}_f + \boldsymbol{C}_1) - \boldsymbol{C}] - \boldsymbol{C}_2 \tag{4f}$$

Summarily, the most general motion forms of cam mechanisms can be boiled down to the following: cam and follower rotate or oscillate and translate. Such motion forms will cover all kinds of the motions described previously, and so the unified vector expression for the cam curved faces or the curves can be constructed as

$$\boldsymbol{R}_c = E^{j\theta_2}[(E^{-k\theta_1}\boldsymbol{R}_f + \boldsymbol{C}_1) - \boldsymbol{C}] - \boldsymbol{C}_2 \tag{4}$$

As the kinematical parameters will be given the different values according to the corresponding

motion forms, Equation (4) can be converted to the previous expression of $\boldsymbol{R}_c$. For example, as in cam or follower without translating motion, h_2 or h_1 will equal 0, so $\boldsymbol{C}_2$ or $\boldsymbol{C}_1$ becomes a zero vector. In case of cam or follower without rotary motion, θ_1 or θ_2 will equal 0, so $E^{j\theta_2}$ or $E^{-k\theta_1}$ can be turned into an identity matrix. In this situation, either the zero vectors or the identity matrix has no influence on the $\boldsymbol{R}_c$ specified by Equation (4). It follows that Equation (4) can represent the cam curved surfaces or the curves of cam mechanisms with a number of motion forms.

In relation to the planar mechanism, since $\boldsymbol{R}_c$ lies in the x-y plane, the rotation matrix will become

$$\begin{cases} E^{j\theta_2} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{bmatrix} \\ E^{-k\theta_1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \end{cases} \tag{5}$$

if $\theta_1 = \theta_2 = 0$, then

$$E^{j\theta_2} = E^{-k\theta_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d)Contact conditions

In addition, since the cam and the follower must satisfy the contact condition at the mating point P, the unified mathematical formula expressed by Equation (4) remains incomplete. According to the meshing theory, a fixed position on the follower is in contact with the corresponding counterpart on the cam at each time t, which indicates that θ_2, θ_1, h_2 and h_1 are all the function of t. From the envelope theory, it is known that the cam-curved surface is a single parameter envelope surface of the curved face of follower with respect to t, and the contact conditions (contact equations) are:

$\phi(\delta_f, \varphi, t) = 0$ for spatial mechanisms

$\phi(\varphi, t) = 0$ for planar mechanisms

that is

$$\phi(\delta_f, \varphi, t) = [\frac{\partial \boldsymbol{R}_c}{\partial \delta_f}, \frac{\partial \boldsymbol{R}_c}{\partial \varphi}, \frac{\partial \boldsymbol{R}_c}{\partial t}] = 0 \tag{6a}$$

The mixing product of the three vectors $\dfrac{\partial \boldsymbol{R}_c}{\partial \delta_f}, \dfrac{\partial \boldsymbol{R}_c}{\partial \varphi}, \dfrac{\partial \boldsymbol{R}_c}{\partial t}$ is zero.

$$\phi(\varphi, t) = \left| \frac{\partial \boldsymbol{R}_c}{\partial \varphi} \times \frac{\partial \boldsymbol{R}_c}{\partial t} \right| = 0 \tag{6b}$$

where

$$\begin{cases} \dfrac{\partial \boldsymbol{R}_c}{\partial \delta_f} = E^{j\theta_2} E^{-k\theta_1} \dfrac{\partial \boldsymbol{R}_f}{\partial \delta_f} \\ \dfrac{\partial \boldsymbol{R}_c}{\partial \varphi} = E^{j\theta_2} E^{-k\theta_1} \dfrac{\partial \boldsymbol{R}_f}{\partial \varphi} \\ \dfrac{\partial \boldsymbol{R}_c}{\partial t} = \omega_2 J_2 E^{j\theta_2}[(E^{-k\theta_1} \mathrm{R}_f + \boldsymbol{C}_1) - \boldsymbol{C}] - \boldsymbol{V}_2 + E^{j\theta_2}(-\omega_1 J_3 \mathrm{R}_f + \boldsymbol{V}_1) \end{cases} \tag{7}$$

In Equation (7), ω_1 and ω_2 are the angular velocities of follower and cam, $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ are the translating velocities of follower and cam, respectively, ω_1, ω_2, $\boldsymbol{V}_1$, $\boldsymbol{V}_2$ embodying the motion specifications of cam mechanisms, are known, J_2 and J_3 are the 2nd and the 3rd component [4] of the rotation matrix.

Hence, the unified mathematical model for the cam curved surfaces and the curve is completely formulated:

$$\boldsymbol{R}_c = E^{j\theta_2}[(E^{-k\theta_1} \boldsymbol{R}_f + \boldsymbol{C}_1) - \boldsymbol{C}] - \boldsymbol{C}_2 \tag{$*$}$$

$$\phi(\delta_f, \varphi, t) = 0 \; or \; \phi(\varphi, t) = 0$$

The contact conditions reflect the relationship between the curved veins coordinates of cam working curved surface at the contact point P. For each time t and any given δ_f, φ can be specified. For a planar cam, when δ_f is known as a constant and φ is eliminated, the curve equation for cam, with t being a parametric variable, can be obtained.

Having formulated the unified mathematical model（∗）for cam profile or working curved surface, the unified formula for the pressure angle and the curvature, etc., which will no longer be discussed here, can be accordingly achieved.

Normalization of the initial parameter for CAD of CAM mechanisms

As the cam profile (or curved surface) is generated, in line with the unified mathematical model developed above, the normalization of the initial parameters must be taken into account to adapt for the diverse cam mechanisms. In other words, how to select and enter the initial parameters is a prerequisite consideration to automatically define every vector and matrix in Equation (∗) by computer. For these vectors and matrices, in addition to the previous expressions of $\boldsymbol{R}_f$, $E^{j\theta_1}, E^{-k\theta_2}$ we can select, the normalizing model for C, C_1, C_2 is described as follows again:

$$C = [\rho_1, \rho_2, -\rho_3]^T \tag{8}$$

where ρ_1, ρ_2, and $-\rho_3$ in Figure 1 are the components of O_1O_2 in the direction of x_1, y_1 and z_1, respectively.

$$C_1 = [h_{1x}, h_{1y}, h_{1z}]^T \tag{9}$$

where h_{1x}, h_{1y} and h_{1z} are components of any follower displacement h_1 in space in the direction x_f, y_f and z_f, respectively. In general, the coordinate system can be so set that the follower moves along one of the axes. Consequently, the other components will become zero.

$$C_2 = [h_{2x}, h_{2y}, h_{2z}]^T \tag{10}$$

where h_{2x}, h_{2y} and h_{2z} are the components of any cam displacement h_2 in the direction of x_2, y_2 and z_2, respectively.

Apparently, the initial parameters in the design of cam profile are involved in $\boldsymbol{R}_f, E^{j\theta_2}, E^{-k\theta_1}, C$, C_1, C_2, where the geometric parameters consist of r_f, r_0 and b as well as ρ_1, ρ_2 and ρ_3; while the kinematical parameters contain θ_2, h_2, θ_1 and h_1. Among them, θ_2 and h_2 are the prescribed kinematical parameters of cam .The kinematical parameters θ_1 and h_1 of follower can be achieved by means of the motion specifications designed .It is assumed that the counterclockwise θ_1 and θ_2 are positive and the initial displacements are covered in four displacement parameters.

The following are several examples for the typical cam mechanisms to illustrate the way in which the initial parameters are chosen in table 1. Only the fixed coordinate systems are plotted for the simplification of the schematic diagrams. Moreover, the selection of $\boldsymbol{R}_f, E^{j\theta_2}, E^{-k\theta_1}$ stated previously will not be repeated any more, and only the axes represented by the j and k axes for the spatial mechanisms will be shown.
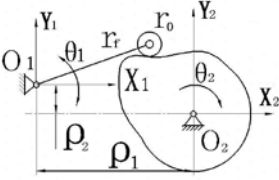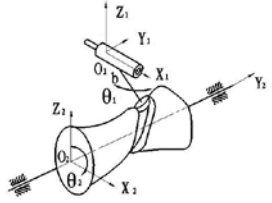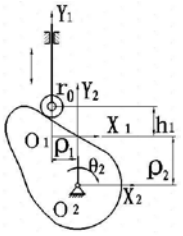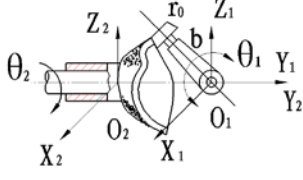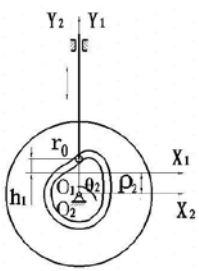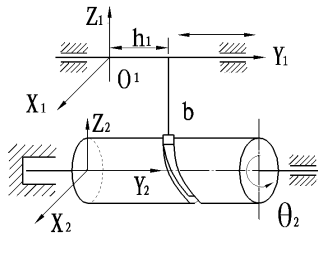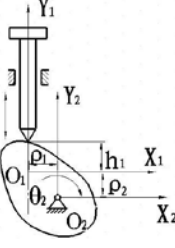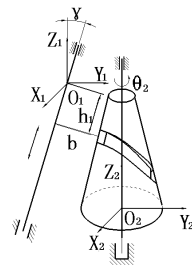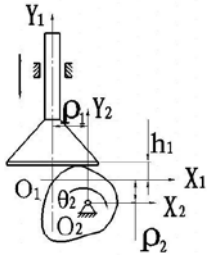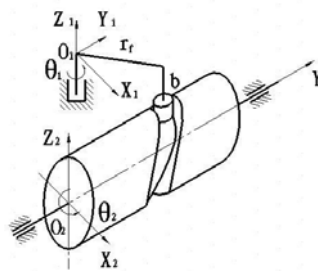
Conclusions

The unified mathematical model presented in this paper is adaptable for the design of cam profile and the geometric analysis of all types of cam mechanisms. In the meanwhile it can be used to develop the general purpose CAD software for the design and the geometric analysis of cam mechanisms. The diverse cam mechanisms, as long as the initial parameters are adequately chosen, can be well manipulated via the unified mathematical model.

Table 1 Typical cam mechanisms name and the initial parameters are chosen

Cam mechanisms figure	Mechanisms name and initial parameters chosen	Cam mechanisms figure	Mechanisms name and initial parameters chosen
	Plate cam - oscillating follower $b = 0, \beta = 0$ $\delta_f = 0$ $C = [\rho_1, -\rho_2]^T$ $C_1 = C_2 = 0$		Globoid cam - oscillating follower $j \to y_2$, $k \to x_1$ $r_f = 0, \beta = 90°$ $C = [0, -\rho_2, -\rho_3]^T$, $C_1 = C_2 = 0$
	Plate cam - translating follower $\theta_1 = 0$, $r_f = 0$ $b = 0$, $\beta = 0$ $\delta_f = 0$ $C = [\rho_1, -\rho_{2]}^T$ $C_1 = [0, h_1]^T$ $C_2 = 0$		End cam - oscillating follower $j \to y_2$, $k \to x_1$ $r_f = 0, \beta = 90°$ $C = [0, -\rho_2, 0]^T$ $C_1 = C_2 = 0$
	Grooved cam - in-line follower $\theta_1 = 0$, $r_f = 0$ $b = 0$, $\beta = 0$ $\delta_f = 0$, $C = [0, -\rho_{2]}^T$ $C_1 = [0, h_1]^T$, $C_2 = 0$ Since this cam possesses the two profiles, so the value of φ will get "+, -" from $\phi(\varphi, t)=0$		Cylindrical cam - translating follower $\theta_2 = 0, j \to y_2$ $\beta = 90°$ $C = [0, \rho_2, -\rho_3]^T$ $C_1 = [0, h_1, 0]^T$ $C_2 = 0$
	Plate cam - knife-edge follower $\theta_1 = 0$, $r_f = 0$ $b = 0, r_0 = 0, \delta_f = 0$ $R_f = 0$ $C = [\rho_1, -\rho_2]^T$ $C_1 = [0, -h_1]^T$, $C_2 = 0$		Conic cam - translating follower $\theta_1 = 0, j \to z_2$ $\beta = \gamma$ $C = [0, \rho_2, -\rho_3]^T$ $C_1 = [0, h_1 \sin\gamma, h_1 \cos\gamma]^T$ $C_2 = 0$
	Plate cam - flat-faced follower $\theta_1 = 0$, $r_f = 0$ $b = 0$ $\delta_{fx} = 0$ $C = [\rho_1, -\rho_2]^T$ $C_1 = [0, h_1]^T$, $C_2 = 0$		Cylindrical cam - oscillating follower $j \to y_2$, $k \to z_1$ $\beta = 90°$ $C = [0, \rho_2, -\rho_3]^T$ $C_1 = C_2 = 0$

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