Phase Synchronization in Small-world Network Composed of Fractional-order Chaotic Oscillator

Feng Chen [1,a], Xiaodan Shao[1] and Long Sheng[2,b]

[1] College of Electronics and Information Engineering, Southwest University, Chongqing, 400715, P.R. China

[2] School of information & electrical engineering, Hebei University of Engineering

[a]Fengchen.uestc@gmail.com (F. Chen), [b]shenglong@hebeu.edu.cn

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Abstract. In this paper, we propose a small-world network model composed of fractional-order Rössler oscillators. We investigate the phase synchronization in this fractional-order chaotic oscillator network. Through numerical simulations, it is found that the fractional-order chaotic phase synchronization is dependent on the topology structure and the coupling strength of chaotic oscillators. This study is helpful for our understanding of the phase synchronization in real networks.

Introduction

The study of complex networks has attracted increasing interest in the past decade, fueled, in particular, by the great recent development of communication and information networks. Many studies across a range of scientific disciplines have demonstrated that complex networks pervade not only the social sciences, but also biology and physics [1–4]. The research of complex network has broad application prospects. One type of topology has been particularly useful in providing insights into the implications of complex connectivity: small-world networks, characterized by the presence of shortcuts that link two randomly chosen nodes regardless of the distance between them. A sharp interest on small-world networks began in 1998, when a beautiful paper of Watts and Strogatz [5] was published. A number of publications on the topic is rapidly increasing [6-8]. The dynamics behavior of small world, especially the synchronous behavior, is a hot issue in the study of complex networks [9-11].

Generally, synchronization can be treated as an appearance of some relations between functions of two processes due to interaction. Phase synchronization in coupled chaotic systems is similar to the phase locking of periodic oscillators, where the locking itself is the only concern. For a certain coupling strength, phase locking can be observed for two chaotic oscillators while their amplitudes remains chaotic and weakly correlated [12]. Chaotic phase synchronization has been detected in many laboratory experiments and natural systems such as electrically coupled neurons, extended ecological systems, periodically driven plasma discharge tubes , lasers, circuits, and solar activities [13-18]. Chunguang Li, Qingyun Wang and other researchers have done a lot of research about phase synchronization in networks of coupled oscillators [19, 20].

Up to date, most of the existing works on phase synchronization of networks are based on integer order chaotic systems, while a few studies have addressed the issue of phase synchronization in the fractional order chaotic system. However, the fractional order chaotic systems match the reality of nature and industrial applications. Furthermore, synchronization of fractional order chaotic systems is starting to attract increasing attention due to its potential applications in secure communications and process control. Therefore, this paper introduces the fractional order system in small world network. We investigate the phase synchronization in small-world networks of fractional-order chaotic oscillators. We further show how the coupling strength and small-world topology affect the phase synchronization behaviors.

The rest of the paper is organized as follows. In section 2, we briefly revisit the concept of small-word network. The network model and phase synchronization are discussed in section 3. Simulations results are shown in section 4. Finally, we draw conclusions in section 5.

Small-world Networks

A small-world network is a special case of complex networks with a high degree of local clustering as well as a small average distance. One of the versions of the model (WS) proposed by Watts and Strogatz looks as the following [5].

- Let N nodes be arranged on a circle. Each node connects to K neighboring nodes. The parameters meet the condition $N > K > \ln(N) >> 1$.
- Then each of those links is rewired with a probability p to one of other randomly chosen nodes. One does not allow a node to be coupled to another node more than once, or to couple with itself. Thus, for $p = 0$, it becomes a regular network; for $p = 1$, it becomes a stochastic network.

However, there is a possibility for the WS model to be broken into unconnected clusters. This problem can be resolved by a slight modification of the WS model, suggested by Newman and Watts (NW) [6]. In the NW model, one does not break any connection between any two nearest neighbors. Instead, one adds with probability p a connection between each unconnected pair of vertices. Likewise, one does not allow a node to be coupled with another node more than once, or to couple with itself. For $p = 0$, it reduces to the originally nearest-neighbor coupled network; for $p = 1$, it becomes a globally coupled network.

The Network Model and Phase Synchronization

This paper considers a small-world network model composed of N fractional Rössler chaotic oscillators. The model can be expressed as:

$$\begin{aligned} \frac{d^{\alpha} x_i}{dt^{a}} &= -\omega_i y_i - z_i + \varepsilon \sum_{j=1}^{N} c_{ij}(x_j - x_i), \\ \frac{d^{\alpha} y_i}{dt^{a}} &= \omega_i x_i + a y_i, \\ \frac{d^{\alpha} z_i}{dt^{a}} &= f + z_i(x_i - b). \end{aligned} \tag{1}$$

Where $i = 1, 2, \cdots, N$, α is the fractional order, ε represents the coupling strength, $\omega_i = 1 + \Delta\omega_i$, and $|\Delta\omega_i| < 1$. In this paper, the maximum value of $|\Delta\omega_i|$ is 0.02. When fractional order $\alpha = 0.9$, we set $a = 0.15,\ f = 0.2,\ b = 10$, so as to ensure the system generate chaotic dynamic behaviors. In this situation, the projections of chaotic oscillators on the plane x-y are shown in Fig. 1.

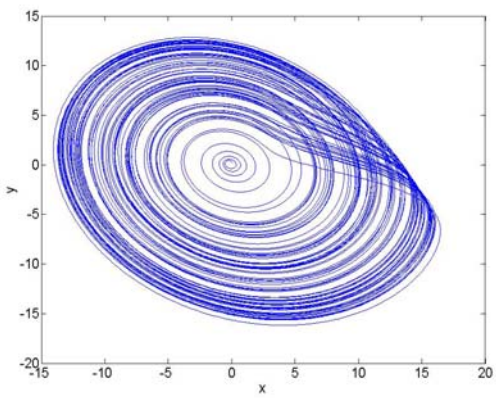

Fig. 1: The phase portrait of the Rössler attractor in the coordinates (x, y).

The matrix $C=\{c_{ij}\}_{N\times N}$ denotes the connection topology, c_{ij} is the connection weights: $c_{ij}=c_{ji}=0$ when chaotic oscillators i and j are not connected; if the chaotic oscillator i is connected to j, $c_{ij}=c_{ji}=1$. The connection weights of chaotic oscillator itself is $c_{ii}=0$.

According to the NW small-world network model described in the section2, network model expressed in (1) will evolve according to the following rule:

- By setting the $C=\{c_{ij}\}_{N\times N}$, the network becomes the originally nearest-neighbor coupled network.
- If $c_{ij}=0$, then replace it by $c_{ij}=1$ with probability p. Setting connection probability p reasonablely, chaotic attractors can couple a small-world network.

For the Rössler chaotic oscillator, because of its simple form, the phases and the amplitudes can be expressed as:

$$\phi_i=\arctan\frac{y_i}{x_i},\ A_i=\sqrt{x_i^2+y_i^2}. \tag{2}$$

Where arctan make phase is continuous in time domain. The mean phase difference of the network is defined as:

$$h(t)=\frac{1}{N-1}\sum_{j=2}^{N}\left|\phi_j(t)-\phi_1(t)\right|. \tag{3}$$

The mean frequency difference of the network is defined as:

$$\Delta\Omega=\frac{1}{N-1}\sum_{j=2}^{N}\left|\langle\dot{\phi}_j(t)-\dot{\phi}_1(t)\rangle\right|. \tag{4}$$

where $\langle\cdot\rangle$ denotes the average over time. When $\Delta\Omega\approx 0$, the mean phase difference among the chaotic oscillators do not grow with time, that is, $h(t)<\text{const}$. At the same time, if the system output amplitude of the chaotic attractors is not relevant, the network achieves phase synchronization.

Numerical simulations

The Effects of Coupling Strength. We consider a network with $N=200$ nodes, with the link probability $p=0.01$. For $pN<1$, it can't generate a small-world network; for $pN>1$, it can generate a small-world network[21]. So, we only consider cases of $pN>1$ in this paper. Simulations adopt the definitions of Riemann-Liouville fractional order calculus. We get a frequency approximation for the fractional order chaotic oscillator.

The time evolutions of the mean phase differences of different coupling strengths are shown in Fig. 2. When coupling strength ε is set to 0.001, the mean phase difference $h(t)$ always increases with the time; when ε increases from 0.001 to 0.01, the mean phase difference increases slightly; afterwards, further increase the value of coupling strength ($\varepsilon=0.05$) makes the mean phase difference $h(t)$ steady around zero.

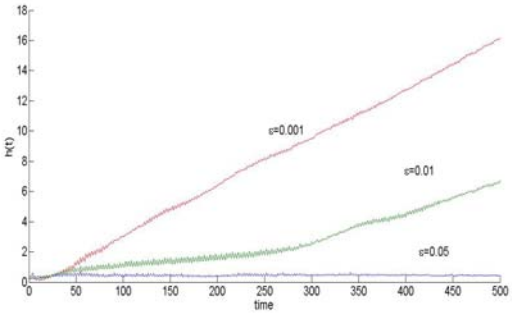

Fig. 2: Time evolutions of mean phase difference $h(t)$

In order to know whether the network model (1) meet the phase synchronization conditions when ε=0.05 , the phase plot of the amplitudes of two randomly selected chaotic oscillators (A_i and A_j) are shown in Fig. 3. We find that when the network is chaotic phase synchronized, its amplitudes remain chaotic. So when the coupling strength ε is 0.05, the output of network model (1) achieves phase synchronization.

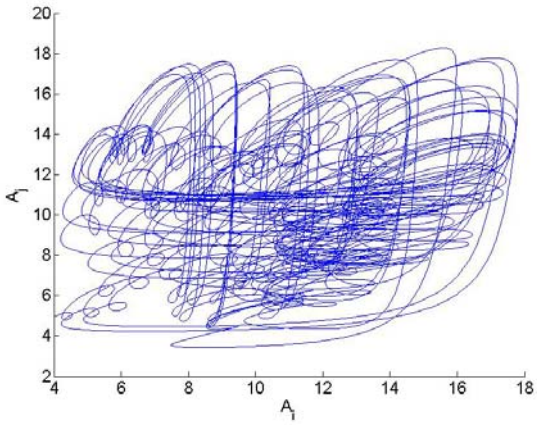

Fig. 3: the phase plot of the amplitudes of two randomly selected chaotic oscillators(ε=0.05).

Furthermore, we study the mean frequency difference $\Delta\Omega$ of network (1) of chaotic oscillators. When the coupling strength increases from 0.005 to 0.06, the change of the mean frequency difference is depicted in Fig. 4. The mean frequency difference $\Delta\Omega$ decreases with the increasing of the coupling strength ε , and the network achieves chaotic phase synchronization when ε passes through the value $\varepsilon = 0.045$.

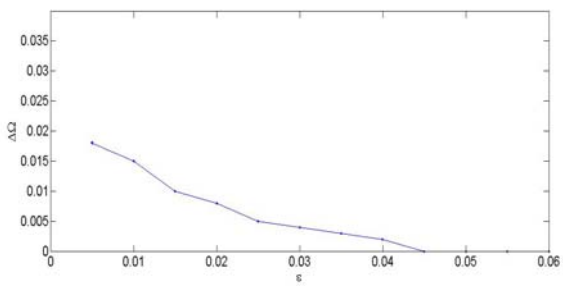

Fig. 4: The mean frequency differences $\Delta\Omega$ as functions of the coupling strength ε .

The Effects of Network Topology. From above sections, we find that the network achieves phase synchronization when the coupling strength exceeds a certain threshold. Due to the aforementioned study, it is interesting to know how the phase synchronization performs over different kinds of network topology. So, in the following simulations, we investigate the behaviors of the network with pN as one parameter, and present representative results from many configurations with different

values of p and N. In Fig. 5, $\varepsilon = 0.01$, the $\Delta\Omega$ decreases with the increasing of the pN, and the network achieves chaotic phase synchronization when pN passes through the value $pN = 18$.

Remark: Although only the results for the fractional order $\alpha = 0.9$ have been presented, the same properties can be observed in the other fractional orders.

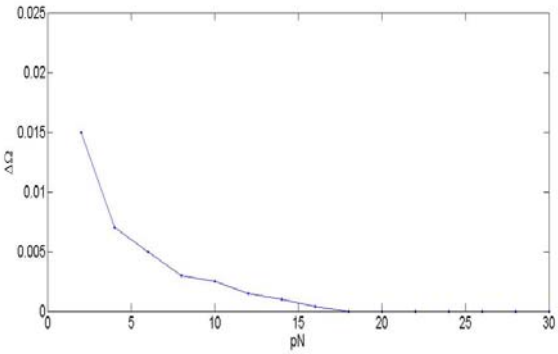

Fig. 5: the product of the probability and the number of chaotic oscillators pN.

Summary

This paper introduces the fractional order system to the small-world network and studies the phase synchronization of small-world fractional-order chaotic oscillator networks. The effects of the coupling strength, topological structure of network for phase synchronization have also been studied. It is found that the phase synchronization can be achieved when the coupling strength or the total number of nonlocal connections per node on average exceeds a certain threshold. The investigation of phase synchronization of small-world fractional-order chaotic oscillator network is realistic and our future work will focus on other more complicated fractional order networks, such as directed networks and scale-free networks.

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