Oscillation of Solutions to Fractional Partial Differential Equations with Several Delays

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***Abstract*—In this paper, we study a class of nonlinear fractional partial differential equations with several delays to the second boundary condition. Based on properties of the Riemann-Liouville fractional derivative, we establish a sufficient oscillatory condition of all solutions. The result is illustrated by an example.**

Keywords-oscillation; fractional partial differential equation; delay

I. INTRODUCTION

Fractional differential equations are generalizations of classical differential equations of integer order. In the last few decades, fractional equations have gained considerable popularity and importance because of their applications in widespread fields of science and engineering, especially in mathematical modeling and simulation of system and processes. Nowadays, some aspects of fractional differential equations, such as the existence, the uniqueness and stability of solutions, the methods for explicit and numerical solutions have been investigated, we refer to [17-20].

In recent years, oscillatory behavior of solutions of fractional ordinary differential equations have been studied by authors [3-11]. However, there is a scarcity in the study of oscillation theory of fractional partial differential equations up to now, we refer to [12-16].

In this article, we are concerned with the oscillation of solutions to the fractional differential equations with several delays of the form

$$D_{+,t}^{1+\alpha}u(t,x)+p(t)D_{+,t}^{\alpha}u(t,x)=a(t)h(u)\Delta u$$

$$+\sum_{i=1}^{m}a_i(t)h_i(u(t-\tau_i(t),x))\Delta u(t-\tau_i(t),x)$$

$$-\sum_{j=1}^{n}q_j(t,x)f_j(u(t-\delta_j(t),x))+g(t,x),$$

$$(t,x)\in R^+\times\Omega\equiv G \qquad (1)$$

with the boundary condition

$$\partial u/\partial n=w(t,x,u),\ (t,x)\in R^+\times\partial\Omega . \qquad (2)$$

Where Ω is a bounded domain in R^n with piecewise smooth boundary $\partial\Omega$; $\alpha\in(0,1)$ is a constant; $G=R^+\times\Omega, R^+=(0,+\infty)$; $D_{+,t}^{\alpha}u$ is the Riemann-Liouville fractional derivative of order α of u with respect of t; Δ is Laplacian operator; and n is the unit exterior normal vector to $\partial\Omega$.

The following conditions are assumed to hold:

A. $a(t),a_i(t),\tau_i(t),\delta_j(t)\in C(R^+,R^+)$; $p(t)\in C(R^+,R)$; $0<\tau_i(t)<\tau, 0<\delta_i(t)<\delta$; δ,τ are constants; $i=1,2,\ldots,m$, $j=1,2,\ldots,n$;

B. $q_j(t,x)\in C(\bar{G};R^+)$; and $q(t)=\min_{1\le j\le n}(\min_{x\in\bar{\Omega}}q_j(t,x))$;

C. $f_j:R\to R$ is a continuous function such that $f_j(u)/u\ge k_j>0$, for all $u\ne 0$, and k_j is a positive constant;

D. $g\in C(\bar{G},R)$;

E. $h(u),h_i(u)\in C(R,R)$; $uh'(u)\ge 0$, $uh_i'(u)\ge 0$; $w(t,x,u)$ is a continuous function, such that $uw(t,x,u)h(u)<0$, $uw(t,x,u)h_i(u)<0$.

By a solution of the problem (1)-(2), we mean a function $u(t,x)$ which satisfies (1) on $\bar{G}$ and boundary condition (2).

A solution $u(t,x)$ of the problem (1)-(2) is said to be oscillatory in G if it is neither eventually positive nor eventually negative, otherwise it is non-oscillatory.

II. PRELIMINARIES AND LEMMAS

A. *Definition 1*

The Riemann-Liouville fractional partial derivative of order $\alpha > 0$ *with respect to* t *of a function* $u(t,x)$ *is given by*

$$D_+^{\alpha} u(t,x) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-v)^{-\alpha} u(v,x)\,dv\,,\ t>0\,, \quad (3)$$

provided the right hand side is pointwise defined on R^+*, Where* Γ *is gamma function.*

B. *Definition 2*

The Riemann-Liouville fractional integral of order $\alpha > 0$ *of a function* $y: R^+ \to R$ *on the half-axis* R^+ *is defined by*

$$(I_+^{\alpha} y)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-v)^{\alpha-1} y(v)\,dv\,,\ t>0\,, \quad (4)$$

provided the right hand side is pointwise defined on R^+.

C. *Definition 3*

The Riemann-Liouville fractional derivative of order $\alpha > 0$ *of a function* x *on the half-axis* R^+ *is defined by*

$$D_+^{\alpha} x(t) := \frac{d^{\lceil \alpha \rceil}}{dt^{\lceil \alpha \rceil}} I_+^{\lceil \alpha \rceil - \alpha} x(t)$$

$$= \frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \frac{d^{\lceil \alpha \rceil}}{dt^{\lceil \alpha \rceil}} \int_0^t (t-v)^{\lceil \alpha \rceil - \alpha - 1} x(v)\,dv\,,$$

$$t > 0\,, \quad (5)$$

provided the right hand side is pointwise defined on R^+. *Where* $\lceil \alpha \rceil$ *is the ceiling function of* α.

D. *Lemma 1*

[2] Let $0<\alpha<1$ *and* $(I_+^{1-\alpha} y)(t)$ *be the fractio-nal integral (4) of order* $1-\alpha$*, then*

$$(I_+^{\alpha} D_+^{\alpha} y)(t) = y(t) - \frac{(I_+^{1-\alpha} y)(0)}{\Gamma(\alpha)} t^{\alpha-1} \quad (6)$$

E. *Lemma 2*

[2]Let $0<\alpha<1$*,* $m \in N$ *and* $D = d/dx$*. If the fractional derivatives* $(D_+^{\alpha} y)(x)$ *and* $(D_+^{m+\alpha} y)(x)$ *exist, then*

$$(D^m D_+^{\alpha} y)(x) = (D_+^{m+\alpha} y)(x) \quad (7)$$

For the sake of convenience, in this article, we denote:

$$U_1(t) = \int_{\Omega} u(t,x)\,dx, \quad G_1(t) = \int_{\Omega} g(t,x)\,dx,$$

$$V(t) = \exp \int_{t_0}^{t} p(\xi)\,d\xi\,, \quad (8)$$

III. MAIN RESULT

A. *Theorem*

Suppose that

$$\lim_{t\to 0} I_+^{1-\alpha} U_1(t) = C_1 \quad (9)$$

where C_1 *is a constant. If*

$$\liminf_{t\to\infty} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} G_1(s)V(s)\,ds\right) d\xi < 0\,, \quad (10)$$

$$\limsup_{t\to\infty} \int_0^t \frac{(t-\xi)^{\alpha-1}}{V(\xi)} \left(C + \int_{t_0}^{\xi} G_1(s)V(s)\,ds\right) d\xi > 0\,, \quad (11)$$

then every solution of the problem (1)-(2) is oscillatory in G. Where C is a constant.

Proof. Suppose to the contrary that there is a non-oscillatory solution $u(t,x)$ of the problem (1)-(2). Without loss of generality, we assume that there exists $T>0$, $t_0>T$, such that $u(t,x)>0$, for all $t \ge t_0$ and $u(t-\tau_i(t),x)>0$, $u(t-\delta_j(t),x)>0$, $i=1,2,\cdots,m$, $j=1,2,\cdots,n$.

Integrating (1) with respect to x over the domain Ω, we get

$$D\left(\int_{\Omega} D_{+,t}^{\alpha} u(t,x)\,dx\right) + p(t)\int_{\Omega} D_{+,t}^{\alpha} u(t,x)\,dx$$

$$= a(t)\int_{\Omega} h(u)\Delta u\,dx$$

$$+\sum_{i=1}^{m} a_i(t)\int_\Omega h(u)(t-\tau_i(t),x)\Delta u(t-\tau_i(t),x)dx$$

$$-\sum_{j=1}^{n}\int_\Omega q_j(t,x) f_j(u(t-\delta_j(t),x))dx+\int_\Omega g(t,x)dx,$$

$$t > t_0, \tag{12}$$

Using Green's formula, boundary condition (2) and *E* yield

$$\int_\Omega h(u)\Delta u dx=\int_{\partial\Omega} h(u)\frac{\partial u}{\partial n}ds-\int_\Omega h'(u)\left|gradu\right|^2 dx$$
$$=\int_{\partial\Omega} h(u)w(t,x,u)ds-\int_\Omega h'(u)\left|gradu\right|^2 dx\le 0, \tag{13}$$

$$\int_\Omega h_i(u(t-\tau_i(t),x))\Delta u(t-\tau_i(t),x)dx\le 0. \tag{14}$$

From *B and C*, we can easily obtain

$$\sum_{j=1}^{n}\int_\Omega q_j(t,x) f_j(u(t-\delta_j(t),x))dx$$

$$\ge\sum_{j=1}^{n} q(t)\int_\Omega f_j(u(t-\delta_j(t),x))dx$$

$$\ge\sum_{j=1}^{n} k_j q(t)\int_\Omega u(t-\delta_j(t),x)dx,\quad t\ge t_0. \tag{15}$$

By Lemma 2.5, it follows from (12)-(15) that

$$D_+^{1+\alpha}U_1(t)+p(t)D_+^{\alpha}U_1(t)$$

$$\le -\sum_{j=1}^{n} k_j q_j(t)U_1(t-\delta_j(t))+G_1(t)<G_1(t)$$

$$t\ge t_0. \tag{16}$$

According to (16) we can see that

$$((D_+^{\alpha}U_1(t))V(t))'$$

$$=(D_+^{1+\alpha}U_1(t))V(t)+p(t)(D_+^{\alpha}U_1(t))V(t)$$

$$<G_1(t)V(t),$$

$$t\ge t_0. \tag{17}$$

Integrating both sides of the above inequality from t_0 to t, we get

$$(D_+^{\alpha}U_1(t))V(t)<(D_+^{\alpha}U_1(t_0))V(t_0)+\int_{t_0}^{t} G_1(s)V(s)ds$$
$$=C+\int_{t_0}^{t} G_1(s)V(s)ds. \tag{18}$$

where $C=(D_+^{\alpha}U(t_0))V(t_0)$. From Lemma 2.4 and (18), we have

$$U_1(t)<\frac{I_+^{1-\alpha}U_1(0)}{\Gamma(\alpha)}t^{\alpha-1}$$

$$+I_+^{\alpha}\left(\frac{C}{V(t)}+\frac{1}{V(t)}\int_{t_0}^{t} G_1(s)V(s)ds\right)$$

$$=\frac{C_1}{\Gamma(\alpha)}t^{\alpha-1}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi} G_1(s)V(s)ds\right)d\xi. \tag{19}$$

Taking $t\to\infty$, from (19) and (10) we can obtain

$$\liminf_{t\to\infty} U_1(t)\le\limsup_{t\to\infty}\frac{C_1}{\Gamma(\alpha)}t^{\alpha-1}+\liminf_{t\to\infty}\frac{1}{\Gamma(\alpha)}$$

$$\cdot\int_0^t\frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi} G_1(s)V(s)ds\right)d\xi<0. \tag{20}$$

which contradicts $U_1(t,x)>0$.

On the other hand, we assume that there exists $T>0$, $t_0>T$, such that $u(t,x)<0$ for all $t\ge t_0$, and $u(t-\tau_i(t),x)<0$, $u(t-\delta_j(t),x)<0$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$. We also have (12). Using the similar methods, we can easily obtain

$$U_1(t)>\frac{I_+^{1-\alpha}U_1(0)}{\Gamma(\alpha)}t^{\alpha-1}$$

$$+I_+^{\alpha}\left(\frac{C}{V(t)}+\frac{1}{V(t)}\int_{t_0}^{t}G_1(s)V(s)ds\right)$$

$$=\frac{C_1}{\Gamma(\alpha)}t^{\alpha-1}$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi}G_1(s)V(s)ds\right)d\xi. \quad (21)$$

Taking $t\to\infty$, from (21) and (11) we can obtain

$$\limsup_{t\to\infty}U_1(t)\ge\liminf_{t\to\infty}\frac{C_1}{\Gamma(\alpha)}t^{\alpha-1}+\limsup_{t\to\infty}\frac{1}{\Gamma(\alpha)}$$

$$\cdot\int_0^t\frac{(t-\xi)^{\alpha-1}}{V(\xi)}\left(C+\int_{t_0}^{\xi}G_1(s)V(s)ds\right)d\xi>0. \quad (22)$$

Which contradicts $U_1(t,x)<0$. The proof is completed.

IV. Example

Consider the fractional differential equation

$$\frac{\partial}{\partial t}(D_{+,t}^{1/2}u(t,x))-D_{+,t}^{1/2}u(t,x)$$

$$=u^2(t,x)\Delta u(t,x)+u^2(t-\frac{\pi}{2},x)\Delta u(t-\frac{\pi}{2},x)$$

$$-u(t-\frac{\pi}{2},x)-(x^2+t^2+1)u(t-\frac{2\pi}{3},x)e^{\left[u(t-\frac{2\pi}{3},x)\right]^2}$$

$$+\frac{1}{2}e^t\sin t\sin x,\quad (t,x)\in R^+\times(0,\pi), \quad (23)$$

with the boundary condition

$$\frac{\partial u(t,x)}{\partial n}=-u(t,x),\quad x=0,\pi,t\in R^+. \quad (24)$$

where $\alpha=1/2$, $\Omega=(0,\pi)$, $n=1$, $p(t)=-1$, $a(t)=a_i(t)=1$, $h(u)=h_i(u)=u^2$, $q_1(t,x)=1$, $q_2(t,x)=t^2+x^2+1$, $f_1(u)=u$, $f_2(u)=ue^{u^2}$, $g(t,x)=\frac{1}{2}e^t\sin t\sin x$, $w(t,x,u)=-u$.

It is easy to verify that the conditions *A*-*E* are satisfied, and $V(t)=e^{t_0-t}$, and

$$G_1(t)=\int_\Omega f(t,x)dx=\int_0^\pi e^t\sin t\sin xdx=e^t\sin t.$$

Hence

$$\int_{t_0}^{\xi}G_1(s)V(s)ds=\int_{t_0}^{\xi}(e^s\sin s)e^{t_0-s}ds$$

$$=e^{t_0}(-\cos\xi+\cos t_0). \quad (25)$$

Let $t_0=\pi/2$, we have

$$\int_0^t\frac{(t-\xi)^{-1/2}}{V(\xi)}\left(C+\int_{t_0}^{\xi}G_1(s)V(s)ds\right)d\xi$$

$$=\int_0^t(t-\xi)^{-1/2}e^{\xi-\pi/2}(C-e^{\pi/2}\cos\xi)d\xi. \quad (26)$$

It is easy to verify (10) and (11) hold. Hence all solutions of the problem (23)-(24) oscillate.

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