

# A Linear Time Algorithm for the Line Subgraph Problem in Halin Graphs

Dingjun Lou, Jun Liang and Guanpu Han

School of Data and Computer Science, Sun Yat-sen University, Guangzhou 510006, People's Republic of China

**Abstract**—Given a graph  $G=(V,E)$  and a positive integer  $p \leq |E|$ . The Line Subgraph Problem is: Is there a subset  $E' \subseteq E$  such that  $|E'| \geq p$  and  $H=(V,E')$  is a line graph. In this paper, we design a linear time algorithm to solve the line subgraph problem for Halin graphs. The algorithm is optimal.

**keywords**-halin graph; line graph; linear time algorithm

## I. INTRODUCTION AND TERMINOLOGY

A Halin graph  $H$  is defined as follows: First, we embed a tree  $T$  in the plane such that each inner vertex of  $T$  has degree at least 3, then we draw a cycle  $C$  through all leaves of  $T$  to form a planar graph. Then  $H=T \cup C$  is called a Halin graph, where  $T$  is called the characteristic tree of  $H$  and  $C$  is called the accompanying cycle of  $H$ . An example of a Halin graph is shown in Figure 1.

The simplest Halin graphs are wheels. Suppose a Halin graph  $H=T \cup C$  is not a wheel. If  $w$  is an inner vertex of  $T$  such that all neighbors  $v_1, v_2, \dots, v_k$  of  $w$  except one neighbor are leaves of  $T$ , then the induced subgraph  $H[\{w \cup \{v_1, v_2, \dots, v_k\}]$  is called a fan of  $H$  and  $w$  is called the center of the fan (See Figure 1). We define a fan with  $k$  leaves of  $T$  to be a  $k$ -fan. If an inner vertex  $w'$  of  $T$  is not the center of a fan but  $w'$  is adjacent to  $k$  consecutive leaves  $v_1, v_2, \dots, v_k$  of  $T$  on  $C$ , then  $H[\{w' \cup \{v_1, v_2, \dots, v_k\}]$  is called a pseudo  $k$ -fan, in particular, if  $k=1$ , then the pseudo  $k$ -fan is called a pseudo single fan.

For all terminology and notation not defined in this paper, the reader is referred to [1].

Halin graphs were introduced by German mathematician Halin [3] as minimally 3-connected planar graphs. It can be used as a model of a network with minimum cost and fault tolerance. Halin graphs have many good properties. For example, many NP-complete or NP-hard problems restricted on Halin graphs can be solved in polynomial time or even in linear time.

The TSP problem for general graphs is an NP-hard problem. However, Cornuejols, Naddef and Pulleyblank [2] give a linear time algorithm to solve the TSP for a weighted Halin graph. Lou [5] proves that each Halin graph is Hamiltonian connected, that is, for each pair of vertices  $u$  and  $v$  in a Halin graph  $H$ , there is a Hamiltonian path from  $u$  to  $v$  in  $H$ . Then Li, Lou and Lu [4] design a linear time algorithm

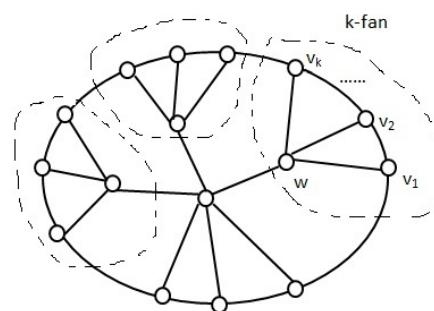


FIGURE 1. AN EXAMPLE FOR A HALIN GRAPH

to find a Hamiltonian path with minimum weight between each pair of vertices in a weighted Halin graph.

Phillips, Punnen and Kabadi [7] design a linear time algorithm to solve the Bottleneck TSP in a weighted Halin graph. Lou and Liu [6] give a linear time algorithm to solve the cubic subgraph problem for Halin graphs, which is to find a cubic subgraph with maximum number of edges in a Halin graph. The problems mentioned above for general graphs are all NP-complete or NP-hard.

In this paper, we give a linear time algorithm to solve the line subgraph problem for Halin graphs. The line subgraph problem is: Given a graph  $G=(V,E)$  and a positive integer  $p \leq |E|$ . Is there a subset  $E' \subseteq E$  such that  $|E'| \geq p$  and  $H=(V,E')$  is a line graph.

The line subgraph problem for general graphs is also NP-complete. We restrict the problem on Halin graphs and design a linear time algorithm to solve the problem, the algorithm is optimal.

The line graph  $H$  of a graph  $G=(V,E)$  is such a graph that, for each edge  $e \in E(G)$ ,  $H$  has a vertex  $v_e$  corresponding to  $e$ , two vertices  $v_e$  and  $v_f$  of  $H$  are adjacent if and only if their corresponding edges  $e$  and  $f$  are adjacent in  $G$  (i.e.  $e$  and  $f$  have a common end vertex in  $G$ ). Such a graph  $G$  is called the base graph of  $H$ .

## II. ANALYZING THE STRUCTURE OF A HALIN GRAPH

In the following, we present a property of the line graphs which is a necessary condition for a graph to be a line graph.

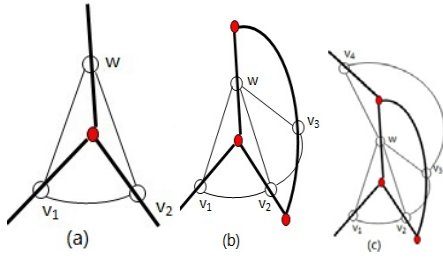


FIGURE II. THE STRUCTURE W IS A FAN

**Lemma 2.1** A line graph is a claw-free graph.

Lemma 2.1 is a well-known property of the line graphs. A claw of a graph  $G$  is an induced subgraph isomorphic to  $K_{1,3}$ .

Now we analyze the structure of a Halin graph  $H=T \cup C$  to find a base graph of a structure  $W$  of  $H$ , where the base graph is drawn with thick lines, and  $W$  is drawn with thin lines.

Case (1): The structure  $W$  is a fan.

Case (1.1): If  $W$  is a 2-fan  $H[\{w\} \cup \{v_1, v_2\}]$ , then its base graph exists (See Figure 2(a)). So 2-fan is a line subgraph.

Case (1.2): If  $W$  is a 3-fan  $H[\{w\} \cup \{v_1, v_2, v_3\}]$ , then its base graph exists (See Figure 2(b)). So 3-fan is a line subgraph.

Case (1.3): If  $W$  is a 4-fan  $H[\{w\} \cup \{v_1, v_2, v_3, v_4\}]$ , then its base graph exists (See Figure 2(c)). So 4-fan is a line subgraph.

Case (1.4): If  $W$  is a  $k$ -fan ( $k \geq 5$ )  $H[\{w\} \cup \{v_1, v_2, \dots, v_k\}]$ , then  $H[\{w\} \cup \{v_1, v_3, v_5\}]$  is a claw, so  $W$  is not a line subgraph, and we shall delete the edges  $wv_5, wv_6, \dots, wv_k$  to find a line subgraph.

Case (2): The center of a fan is adjacent to an inner vertex of  $T$ .

Case (2.1): If the fan  $W$  is a 2-fan, then the center of  $W$  can be adjacent to an inner vertex of  $T$ , the base graph exists (See Figure 3).

Case (2.2): If the fan  $W$  is a  $k$ -fan ( $k \geq 3$ ),  $W=H[\{w\} \cup \{v_1, v_2, \dots, v_k\}]$ , and if  $w$  is adjacent to an inner vertex  $x$  of  $T$ , then  $H[\{w\} \cup \{x, v_1, v_3\}]$  is a claw. So, to find a line subgraph, the center of a  $k$ -fan ( $k \geq 3$ ) cannot be adjacent to any inner vertex of  $T$ .

Case (3): The leaves of  $T$  in a fan  $W$  are adjacent to other vertices on  $C$ .

Case (3.1): If  $W$  is a 2-fan, and  $W=H[\{w\} \cup \{v_1, v_2\}]$ , then  $v_1$  and  $v_2$  can be adjacent to the neighbors on  $C$ , the base graph exists (See Figure 4(a)). In fact,  $v_1$  ( $v_2$ ) can be connected by a path on  $C$  to  $v_5$  ( $v_3$ ) in the line subgraph.

$v_4$ ], then  $v_1$  and  $v_4$  can be adjacent to the neighbors  $v_5$  and  $v_7$  respectively on  $C$  (See Figure 4(c)). In fact,  $v_1$  ( $v_4$ ) can be connected by a path on  $C$  to  $v_5$  ( $v_7$ ) in the line subgraph.

Case (4): An inner vertex  $w$  of  $T$  is contained in pseudo fans.

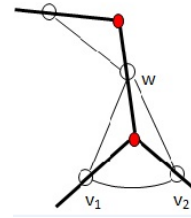


FIGURE III. THE CENTER OF A 2-FAN IS ADJACENT TO AN INNER VERTEX OF  $T$

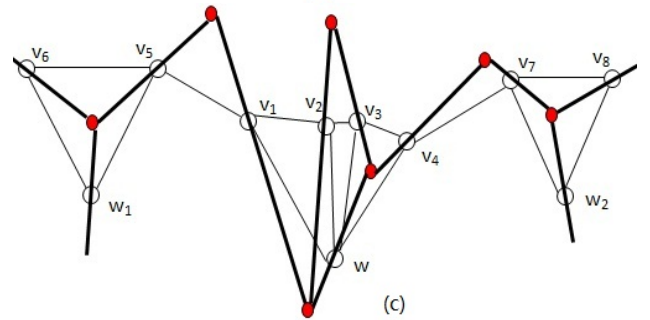
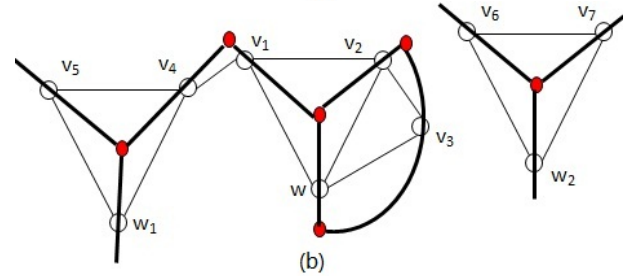
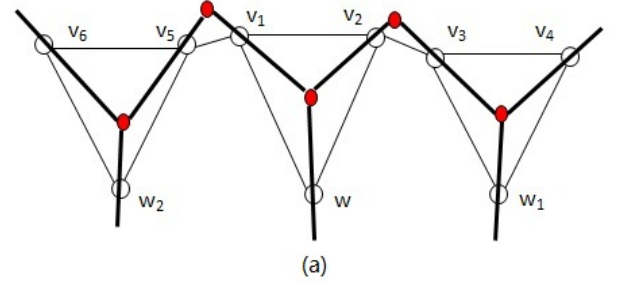


FIGURE IV. THE LEAVES OF  $T$  IN A FAN  $W$  ARE ADJACENT TO OTHER VERTICES ON  $C$

Case (3.2): If  $W$  is a 3-fan, and  $W=H[\{w\} \cup \{v_1, v_2, v_3\}]$ , then  $v_1$  can be adjacent to a neighbor  $v_4$  on  $C$ , but  $v_3$  cannot (See Figure 4(b)). So, to find a line subgraph with maximum number of edges, we must delete one edge  $wv_3$ , then treat the fan  $W-wv_3$  as a 2-fan. Then  $w$  can be adjacent to an inner vertex of  $T$  and  $v_3$  can be adjacent to the neighbor  $v_6$  on  $C$ .

Case (3.3): If  $W$  is a 4-fan, and  $W=H[\{w\} \cup \{v_1, v_2, v_3,$

Case (4.1): If the inner vertex  $w$  of  $T$  is contained in a pseudo single fan  $W=H[\{w\} \cup \{v_1\}]$ , then  $H[\{v_1\} \cup NH(v_1)]$  is a claw. So the edge  $wv_1$  will be deleted to find a line subgraph.

Case (4.2): If the inner vertex  $w$  of  $T$  is contained in one pseudo 2-fan  $W=H[\{w\} \cup \{v_1, v_2\}]$ , similar to Cases(1.1),

(2.1) and (3.1),  $W$  can be kept and  $w$  can be adjacent to an inner vertex of  $T$ , and  $v_1(v_2)$  can be adjacent to its neighbors on  $C$ .

Case (4.3): If the inner vertex  $w$  of  $T$  is contained in one pseudo 3-fan  $W=H[\{w\} \cup \{v_1, v_2, v_3\}]$ , then the edge  $wv_3$  will be deleted,  $w$  can be adjacent to an inner vertex of  $T$  and  $v_1(v_3)$  can be adjacent to its neighbors on  $C$ .

Case (4.4): If the inner vertex  $w$  of  $T$  is contained in one pseudo 4-fan  $W=H[\{w\} \cup \{v_1, v_2, v_3, v_4\}]$ , similar to Cases(1.3), (2.2) and (3.3),  $W$  can be kept,  $w$  cannot be adjacent to any inner vertex of  $T$ , and  $v_1(v_4)$  can be adjacent to its neighbors on  $C$ .

Case (4.5): If the inner vertex  $w$  of  $T$  is contained in one pseudo  $k$ -fan  $W=H[\{w\} \cup \{v_1, v_2, \dots, v_k\}]$  for  $k \geq 5$ , similar to Cases(1.4), (2.2) and (3.3),  $W$  is not a line subgraph, we shall delete the edges  $wv_5, wv_6, \dots, wv_k$ ,  $w$  cannot be adjacent to any inner vertex of  $T$  and  $v_1(v_k)$  can be adjacent to its neighbors on  $C$ .

Case (4.6): If the inner vertex  $w$  of  $T$  is contained in at least two pseudo fans  $W_1=H[\{w\} \cup \{v_1, v_2, \dots, v_r\}]$ ,  $W_2=H[\{w\} \cup \{u_1, u_2, \dots, u_s\}]$  ( $r \geq 2$  and  $s \geq 2$ ) and  $W_3, W_4, \dots, W_t$ , then delete the edges  $wv_3, wv_4, \dots, wv_r, wu_3, wu_4, \dots, wu_s$  and all edges from  $w$  to the leaves of  $T$  in  $W_3, W_4, \dots, W_t$ , and  $w$  cannot be adjacent to any inner vertex of  $T$  and the edges on  $C$  incident with (or contained in)  $W_1, W_2, \dots, W_t$  remain. The remaining graph is a line subgraph (See Figure 5).

Case (5): The structure of inner vertices of  $T$ .

If  $w$  is an inner vertex of  $T$  such that  $w$  is not the center of a fan or a pseudo fan,  $d_T(w) \geq 3$ , and the vertices in  $NH(w)$  are inner vertices of  $T$ , then  $H[\{w\} \cup NH(w)]$  forms a claw. So the inner vertices of  $T$  form a line subgraph if and only if they form a forest with each tree to be a path by deleting some edges of  $T$ .

To obtain a line subgraph with maximum number of edges in  $T-S$  ( $S$  is the set of all leaves of  $T$ ), the inner vertices of  $T$  form a spanning subgraph  $T'$  of  $T-S$ , which is a set of paths.  $T'$  satisfies the following properties: The vertices  $x_1, x_2, \dots, x_r$  of degree less than 2 in  $T'$  are adjacent (in  $T$ ) to vertices  $y_1, y_2, \dots, y_s$  of degree 2 (in  $T'$ ) but  $x_1, x_2, \dots, x_r$  are not adjacent to  $y_1, y_2, \dots, y_s$  in  $T'$ ; or  $x_1, x_2, \dots, x_r$  are adjacent (in  $T$ ) to vertices  $w_1, w_2, \dots, w_t$ , where  $w_1, w_2, \dots, w_t$  are centers of  $k$ -fans or pseudo  $k$ -fans, but by Cases (2.2), (3.2) and (4),  $w_1, w_2, \dots, w_t$  are not adjacent to  $x_1, x_2, \dots, x_r$  in  $T'$ ; or  $x_i$  belongs to  $\{w_1, w_2, \dots, w_t\}$  for some  $i$ 's, and by Cases (2.2), (3.2) and (4),  $x_i$  has degree less than 2 in  $T'$ .

To see that  $T'$  has maximum number of edges, suppose another spanning line subgraph  $T''$  has maximum number of edges, but  $x_i$  ( $1 \leq i \leq r$ ) has degree  $d_{T'}(x_i) + 1$  in  $T''$ , then  $x_i y_j$  belongs to  $E(T'')$ , but then  $y_j$  has degree at least 3, so at least one edge incident with  $y_j$  (in  $T'$ ) must be deleted in  $T''$ ; or  $x_i w_j$

belongs to  $E(T'')$ , but by Cases (2.2), (3.2) and (4),  $w_j$  cannot have degree more than 1, so the edge incident with  $w_j$  in  $T'$  must be deleted in  $T''$ ; or  $x_i$  belongs to  $\{w_1, w_2, \dots, w_t\}$ , by Cases (2.2), (3.2) and (4),  $x_i$  cannot have degree more

than its original degree in  $T'$ , otherwise the above two cases happen. We can obtain  $T''$  from  $T'$  by a series of operations as above three cases. We can see that  $T''$  will not have more edges than  $T'$ .

### III. ALGORITHM TO FIND A MAXIMUM LINE SUBGRAPH

For any given Halin graph  $H=T \cup C$ , we use the following algorithm to find a maximum line subgraph for  $H$ .

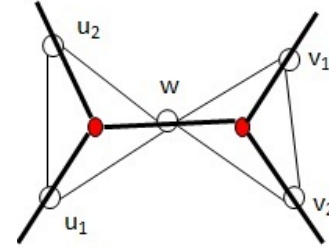


FIGURE V. THE INNER VERTEX  $w$  OF  $T$  IS CONTAINED IN AT LEAST TWO PSEUDO  $k$ -FANS

*Algorithm:*

1. We do postorder traversal of  $T$  and choose an inner vertex  $r$  of  $T$  as the root such that  $r$  is not the center of any fan if there is such a vertex; otherwise  $r$  is any inner vertex of  $T$ .

2. When we visit an inner vertex  $w$  of  $T$  in the postorder traversal,

(2.1) If  $w$  is the center of a wheel  $W=H[\{w\} \cup \{v_1, v_2, \dots, v_k\}] = H$ , and if  $k \geq 4$ , then delete  $wv_5, wv_6, \dots, wv_k$  from  $H$ ; if  $k=3$ , then delete  $wv_3$ . The resulting graph is a maximum line subgraph.

(2.2) If  $w$  is the center of a  $k$ -fan  $W=H[\{w\} \cup \{v_1, v_2, \dots, v_k\}]$ ,

(2.2.1) If  $k=2$ , keep the edge joining  $w$  to another inner vertex in  $T$ ;

(2.2.2) If  $k=3$ , delete the edge  $wv_3$ , and keep the edge joining  $w$  to another inner vertex in  $T$ ;

(2.2.3) If  $k=4$ , delete the edge joining  $w$  to another inner vertex of  $T$ ;

(2.2.4) If  $k \geq 5$ , delete the edges  $wv_5, wv_6, \dots, wv_k$  and the edge joining  $w$  to another inner vertex of  $T$ ;

(2.3) If  $w$  is contained in a pseudo  $k$ -fan  $W_1=H[\{w\} \cup \{v_1, v_2, \dots, v_k\}]$ ,

(2.3.1) While  $w$  is contained in a pseudo single fan  $W_1=H[\{w\} \cup \{v_1\}]$ , do deleting the edge  $wv_1$ ;

(2.3.2) If  $w$  is contained in one pseudo  $k$ -fan  $W_1$ , then

(2.3.2.1) If  $k=2$ , keep only one edge which joins  $w$  to another inner vertex as its child in  $T$ ; if there is not such a child, then keep the edge to its father; Then delete the other edges joining  $w$  to other inner vertices of  $T$ ;

(2.3.2.2) If  $k=3$ , delete  $wv_3$ , then do the same work as (2.3.2.1);

(2.3.2.3) If  $k=4$ , delete the edges joining  $w$  to other inner vertices of  $T$ ;

(2.3.2.4) If  $k \geq 5$ , delete the edges  $wv_5, wv_6, \dots, wv_k$  and do the same work as (2.3.2.3).

(2.3.3) If  $w$  is contained in at least two pseudo fans  $W_1=H[\{w\} \cup \{v_1, v_2, \dots, v_r\}]$ ,  $W_2=H[\{w\} \cup \{u_1, u_2, \dots, u_s\}]$  and  $W_3, W_4, \dots, W_t$ , and  $r, s \geq 2$ , then delete  $wv_3, wv_4, \dots, wv_r$ ;  $wu_3, wu_4, \dots, wu_s$ ; and all edges joining  $w$  to the leaves of  $T$  in  $W_3, W_4, \dots, W_t$ . Then delete the edges joining  $w$  to other inner vertices of  $T$ .

(2.4) If  $w$  is only adjacent to inner vertices of  $T$ , then two edges to its children are kept; if  $w$  is adjacent to only one child, then we keep one edge to its child and one edge to its father. Then we delete the other edges incident with  $w$ . If there is only one edge incident with  $w$ , we keep it.

3. When we have visited the root  $r$ , all vertices have been processed as in Step 2, and then the resulting graph is a maximum line subgraph.

By the analysis of the structure of a Halin graph  $H$  in Section 2, we can see that our algorithm is correct. Since our algorithm only do once postorder traversal of  $T$ , the worst time complexity of the algorithm is  $O(n)$ , where  $n$  is the number of vertices in  $H$  (we shall also notice that  $|E(H)|=|V(H)|=O(n)$ ).

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