Study on Capacity Expansion Problem of Minimum Cost Flow

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Abstract. The minimum cost flow problem is the central object of study in the network flows. This paper studies capacity expansion problem of minimum cost flow. How to expand the network capacity effectively so that the network flow can reach a certain level with minimum total cost? In view of three types of expanding models: the arc-expanding model, the node-expanding model and the combination of arc-node expanding model; this article discusses the characteristics of the problems separately and a unified capacity expansion model is also proposed. Finally, an example has been provided in detail.

Introduction and Motivation

Capacity expansion can be seen almost everywhere in the world. For example, in order to increase the flow ability of a road network to a certain level, we wish to expand the road capacity. Similar expansion occurs to transportation system or telecommunication network.

So far, several works have been done. Omar J. Guerra, Diego A. Tejada and Gintaras V. Reklaitis [1] presented a comprehensive optimization framework for the design and planning of interconnected power systems, including the integration of generation and transmission capacity expansion planning. Sinan Gürel [2] considered a multicommodity network flow problem with flow routing and discrete capacity expansion decisions. Miguel Gueifão Santos and António Pais Antunes [3] introduced an optimization model aimed to assist aviation authorities in their strategic decisions regarding the long-term expansion of a network of airports. RL Burdett [4] considered the expansion of railways in order to increase their capacity. Ahuja and Orlin [5] considered how to improve the capacity of network to send flow as much as possible. G.N.Frederickson [6] considered the problems of increasing the weight of minimum spanning tree. C. Yang and G. Liu [7] studied a model for multiperiod capacity expansion problems on networks. L.P. Wang, S. Z. Wang and G.H. Xu [8] introduced a model for the network capacity expansion with the time and cost constraints. C.Yang. and J.Zhang [9] studied the bottleneck capacity expansion problems on networks. J. Yang and C. Yang [10] considered a model of flow capturing location-allocation problem with chance constraint. H. C. Yan, X.H. Huang and M. Wang [11] dealed with the problem of delay-dependent stability for networked control systems with multiple time- delays.

In this article, we assume a directed network G（V，A，C，B，F） consisting of a set of nodes V= $\{v_1, v_{2,} \ldots, v_n\}$, a set of arcs $A \subseteq \{(v_i, v_j) \mid i = 1, 2, \cdots, n, j = 1, 2, \cdots, n\}$, a capacity vector C, a cost vector B and a network flow vector F. The component c_{ij} of C denotes the capacity of arc (v_i, v_j). The component b_{ij} of B denotes the unit cost of arc (v_i, v_j). The component f_{ij} of F denotes the flow of arc (v_i, v_j). The source node is v_s and sink node is v_t.

There Are Three Types of Expanding Models. The Arc-Expanding Model. In this model, the unit cost to increase the capacity of arc (v_i, v_j) is ω_{ij} .The network capacity is increased by the expansion of arcs. This model is widely used. The path widening belongs to this kind of expanding model.

The Node-Expanding Model. On this model, it is assumed that the capacities of all arcs

(v_i, v_j) which start at the same node v_i should be increased by the same amount and that the unit cost to make such expansion is ω_i. For example, the installation of new concentrators at the nodes of a local access telecommunications network can be viewed as this kind of expanding model.

The Combination of Node-Expanding and Arc-Expanding Model. In the process of network adjustment, the two models may carry on simultaneity. For example, in the telecommunications industry, capacity expansion can be realized through the installation of new concentrators at the nodes and replacing existing links with new type of links.

Problem Formulation

Given a directed network G（V，A，C，B，F）defined as above. Let α_i be the amount for expansion capacity on node v_i, α_{ij} be the amount for expansion capacity on arc（v_i, v_j）, R be a given flow value that the network should be enhanced to, D be a given budget.

We also assume that the capacity and flow on every arc be integers and the optimal solutions be integers, too.

The capacity expansion problem can be stated as follows: for a given value R, how can the flow of minimum cost flow be increased to R with a minimum total cost?

To Solve This Problem, We Will Discuss It on the Three Expanding Models. On the arc-expanding model, the problem can be stated as (LP1):

$$\min\left(\sum_{(v_i,v_j)\in V} b_{ij} f_{ij} + \sum_{(v_i,v_j)\in V} \omega_{ij}\alpha_{ij}\right) \tag{1}$$

$$s.t.\ \sum f_{ij} - \sum f_{ji} = \begin{cases} R, & i = s \\ -R, & i = t \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

$$0 \le f_{ij} \le c_{ij} + \alpha_{ij} \tag{3}$$

$$\alpha_{ij} \ge 0,\ f_{ij} \ge 0 \tag{4}$$

This is a linear programming. The objective function states that we want to minimize the sum of the expanding and transportation cost. We refer to the constraints (2) as mass balance flow bound constraints, constraints（3）as flow bound constraints, and constraints (4) as nonnegative constraints.

On the arc-expanding model , the capacity expansion problem can be transformed into a minimum cost flow problem with flow value R in an auxiliary network $G'(V, A^*, C', B')$ as follows: for each arc $(v_i, v_j) \in A$, we have a second arc $(v_i, v_j)'$. Let A' be the set of all second arcs and $A^* = A \cup A'$. The capacity vector C' is defined by

$$c'_{ij} = \begin{cases} +\infty, \forall (v_i, v_j)' \in A'; \\ c_{ij}, \forall (v_i, v_j) \in A \end{cases} \tag{5}$$

And the cost vector is defined by

$$b'_{ij} = \begin{cases} b_{ij}, \forall \left(v_i, v_j\right) \in A; \\ b_{ij} + \omega_{ij,} \forall \left(v_i, v_j\right)' \in A' \end{cases} \tag{6}$$

Now we find out a minimum cost flow f^* from v_s to v_t with a total flow value R in network $G'(V, A^*, C', B')$, which can be written as follows (LP1-1):

$$\min \sum_{\left(v_i, v_j\right)\in A^*} b'_{ij} f_{ij} \tag{7}$$

s.t. $0 \le f_{ij} \le c'_{ij}, \forall (v_i, v_j) \in A^*,$ (8)

$$\sum_j f_{ij} - \sum_j f_{ji} = \begin{cases} R, if\ \ i = s; \\ -R, if\ \ i = t; \\ 0, \text{otherwise} \end{cases} \tag{9}$$

The problem can be solved by strong polynomial algorithms and the optimal flow f_{ij}^* on the arc $(v_i, v_j)'$ corresponds to the capacity value which we need to increase on arc （v_i, v_j）.The total cost for expansion and transportation is the objective value of (LP1-1), that is, the minimum cost of flow f* on $G'(V, A^*, C', W')$. Therefore, the problem (LP1) can be solved by strong polynomial algorithm.

On the node-expanding model, the problem can be expressed as (LP2):

$$\min\left(\sum_{(v_i,v_j)\in V} b_{ij} f_{ij} + \sum_{i=1}^{n} \omega_i \alpha_i\right) \tag{10}$$

$$\sum f_{ij} - \sum f_{ji} = \begin{cases} R, i = s; \\ -R, i = t; \\ 0, \text{othwise} \end{cases} \tag{11}$$

$$0 \le f_{ij} \le \alpha_i + c_{ij} \tag{12}$$

$$\alpha_i \ge 0,\ f_{ij} \ge 0 \tag{13}$$

This is a linear programming. The objective function states that we want to minimize the sum of node-expanding and transportation cost. The constraints (11), (12), (13) are the same as (LP1).

On the combination of node-expansion and arc expansion model, the problem can be stated as (LP3):

$$\min\left(\sum_{(v_i,v_j)\in V} b_{ij} f_{ij} + \sum_{(v_i,v_j)\in A} \omega_{ij}\alpha_{ij} + \sum_{v_i\in V} \omega_i \alpha_i\right) \tag{14}$$

$$\sum_{\mathrm{j}} f_{ij} - \sum_{\mathrm{j}} f_{ji} = \begin{cases} R, if\ \ i = s; \\ -R, if\ \ i = t; \\ 0, \text{otherwise} \end{cases} \tag{15}$$

$$0 \le f_{ij} \le \alpha_i + \alpha_{ij} + \underline{c}_{ij} \tag{16}$$

$$\alpha_{ij} \ge 0,\ f_{ij} \ge 0 \tag{17}$$

This is a linear programming. The objective function states that we want to minimize the sum of expanding and transportation cost. The constraints (15), (16), (17) are the same as (LP1).

We can see: (LP1) and (LP2) are all special cases of (LP3). We only consider one type of expanding model on (LP1) and (LP2). That is, the general expanding type is the combination of node-expansion and arc expansion model. So, (LP3) is the general model.

Case Study

Given a directed network G（V，A，C，B，F）described in Fig.1. The first number in brackets besides every arc is the capacity c_{ij} of that arc and the second is the unit cost b_{ij} of that arc. It is easy to know the present maximum flow is 5. The unit cost ω_{ij} for arc-expanding and ω_i for node-expanding are shown in Table 1 and 2 respectively.

The problem is: for a given value R=8, how can the flow of minimum cost flow be enhanced to 8 with minimum total cost?

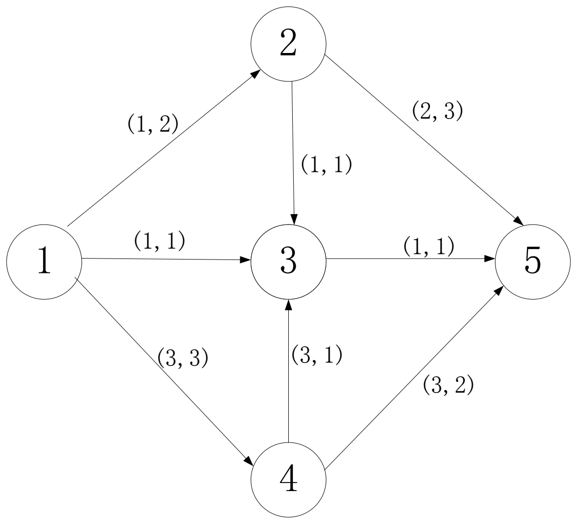

Figure 1. Network for Expansion

Table 1 The unit cost to increase the capacity of arcs

Arcs	(1，2)	(1，3)	(1，4)	(2，3)	(2，5)	(3，5)	(4，3)	(4，5)
ω_{ij}	3	2	4	2	2	3	2	1

Table 2 The unit cost to increase the capacity of nodes

Nodes	1	2	3	4	5
ω_i	1	3	4	1	5

As stated above, we establish mathematical formulations separately. We solve the problem with Excel. The results are shown below.

On the arc-expanding model, the total cost is 43. The expansion arcs are (1, 3) and (3, 5), the expansion capacity are both 3.

On the node-expanding model, the minimum cost is 40. The expansion nodes are 1, 3 and 4, the expansion capacity are all 1.

On the combination of node-expanding and arc-expanding model, the minimum cost is 39. The expanding arc is (3, 5) and expansion capacity is 1. The expansion node is 1 and 4. The expansion capacity are both1.

The results show: on the combination of node-expanding and arc-expanding model, the expansion cost is the lowest among the three models.

Conclusion

Either node-expanding model or arc-expanding one has its own superiority. If they could be combined properly, a better result can be reached. We also point out: the combination of node-expansion and arc expansion model is the general expanding type. Under the foundation of separate discussion, we establish a general linear programming model. This makes the model more practical. Possible extension of this paper would be to allow arbitrary demand. A more general model is currently under investigation.

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