A Sufficient and Necessary Condition for G-expectation to be Linear

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Abstract. In general, Peng's g-expectation is a nonlinear mathematical expectation. However, if the function g (t, x, y) satisfies some properties, the g-expectation is expected to be linear. For g-expectation, we have the following conclusions: The necessary and sufficient condition for g-expectation to have linear properties is that the function g has nothing to do with variable y, and g is linear with variable z. In this case, there exists a probability measure Q, under which the linear expectation is equivalent to Peng's g-expectation.

Introduction

The concept of expectation is clearly very important in probability theory. Expectation is usually defined via

$$E\xi = \int_{-\infty}^{+\infty} x dF(x), \tag{1}$$

Where $F(x) := P(\xi \le x)$ is the distribution of random variable ξ with respect to the probability measure P. One of the properties of mathematical expectation is its linearity: for given random variables ξ and η,

$$E(\xi + \eta) = E\xi + E\eta. \tag{2}$$

This is equivalent to the additivity of probability measure, that is,

$$P(A+B) = P(A) + P(B), \quad \text{If } A \cap B = \Phi. \tag{3}$$

From this viewpoint, we sometimes call mathematical expectation (resp. probability measure) linear mathematical expectation (resp. linear probability measure). It is well known that linear mathematical expectation is a powerful tool for dealing with stochastic phenomena. Economists have found that linear mathematical expectations result in the Allais paradox and the Ellsberg paradox, see Allais [1] and Ellsberg [2].

Peng [3, 4] introduced a kind of nonlinear expectation (he calls it the g-expectation) via a particular nonlinear backward stochastic differential equation (BSDE for short). Some applications of Peng's g-expectation in economics are considered in [5-9]. A question is the following: when is the sufficient and necessary condition for the g-expectation to be linear. We note that Peng's g-expectations can be defined only in a BSDE framework, therefore, we consider the function g.

BSDE and G-expectation

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_s)_{s\ge 0}$, and let $(W_s)_{s\ge 0}$ be a standard d-Brownian motion. For ease of exposition, we assume d = 1. The results of this paper can be easily extended to the case d > 1. Suppose that $(\mathcal{F}_s)$ is the σ-filtration generated by $(W_s)_{s\ge 0}$, that is $\mathcal{F}_t = \sigma\{W_s, 0 \le s \le t\}$.

Let $T > 0$ $\mathcal{F}_T = \mathcal{F}$ and $g = g(y,z,t) : R \times R^d \times [0,T] \to R$ be a function satisfying

(H.1) $\forall (y,z) \in R \times R^d, g(y,z,t)$ is continuous in t and $\int_0^T g^2(0,0,t)dt < \infty$;

(H.2) g is uniformly Lipschitz continuous in (y, z), that is, there exists a constant $C>0$, such that $\forall y,y'\in R, z,z'\in R^d, |g(y,z,t)-g(y',z',t)|\le C(|y-y'|+|z-z'|)$;

(H.3) $g(y,0,t)\equiv 0, \forall (y,t)\in R\times[0,T]$.

Let $\mathrm{M}(0,T,R^n)$ be the set of all square integrable R^n-valued, F_t-adapted processes $\{v_t\}$ with

$$E\int_0^T v_t^2 dt<\infty. \tag{4}$$

For each $t\in[0,T]$, be $L^2(\Omega,\mathrm{F}_t,P)$ the set of all F_t-measurable random variables. Pardoux and Peng [10] considered the following backward stochastic differential equation:

$$y_t=\xi+\int_t^T g(y_s,z_s,s)ds-\int_t^T z_s dW_s, 0\le t\le T, \tag{5}$$

And showed the following result:

Lemma1. Suppose that g satisfies (H.1)–(H.3) and $\xi\in L^2(\Omega,\mathrm{F},P)$. Then Eq. 5 admits a unique solution $(y,z)\in \mathrm{M}(0,T,R)\times \mathrm{M}(0,T,R^d)$.

Using the solution of Eq. 5, Peng [3] introduced the concept of g-expectation via Eq. 5.

Definition1. Suppose g satisfies (H.1)–(H.3). Given $\xi\in L^2(\Omega,\mathrm{F},P)$, let (y,z) be the solution of Eq. 5. We denote Peng's g-expectation of ξ by $\mathrm{E}_g(\xi)$ and define $\mathrm{E}_g(\xi):=y_0$.

The g-expectation $\mathrm{E}_g(\xi)$ preserves many of the properties of classical mathematical expectation. However, it does not preserve linearity. See, for example, Peng [3] for details.

Main Results

For the linearity of g-expectation $\mathrm{E}_g(\xi)$ we have the following result.

Theorem 1 $\forall \xi,\eta\in L^2(\Omega,\mathrm{F}_T,P)$, then $\mathrm{E}_g(\xi+\eta)=\mathrm{E}_g(\xi)+\mathrm{E}_g(\eta)$ holds if and only if g has nothing to do with variable y, and g is linear in variable z.

Proof (Sufficiency). Since g has nothing to do with variable y, and g is linear in variable z, then $\forall (y_i,z_i), i=\{1,2\}$, we have

$$g(s,y_1,z_1)+g(s,y_2,z_2)=g(s,z_1)+g(s,z_2)=g(s,z_1+z_2), \forall \xi,\eta\in L^2(\Omega,\mathrm{F}_T,P) \tag{6}$$

Let
$$\begin{cases} y_t^{\xi}=\xi+\int_t^T g(s,z_s^{\xi})ds-\int_t^T z_s^{\xi}dW_s \\ y_t^{\eta}=\eta+\int_t^T g(s,z_s^{\eta})ds-\int_t^T z_s^{\eta}dW_s \\ y_t^{\xi+\eta}=\xi+\eta+\int_t^T g(s,z_s^{\xi+\eta})ds-\int_t^T z_s^{\xi+\eta}dW_s \end{cases},$$

Then,

$$\begin{aligned} y_t^{\xi}+y_t^{\eta}&=\xi+\eta+\int_t^T g(s,z_s^{\xi})+g(s,z_s^{\eta})ds-\int_t^T z_s^{\xi}+z_s^{\eta}dW_s \\ &=\xi+\eta+\int_t^T g(s,z_s^{\xi}+z_s^{\eta})ds-\int_t^T z_s^{\xi}+z_s^{\eta}dW_s. \end{aligned} \tag{7}$$

Hence, $(y_t^{\xi}+y_t^{\eta}, z_t^{\xi}+z_t^{\eta})$ is a solution of BSDE with terminal state $\xi+\eta$. By the uniqueness of solution of BSDE, $y_t^{\xi+\eta}=y_t^{\xi}+y_t^{\eta}, \forall 0\le t\le T$, and $y_0^{\xi+\eta}=y_0^{\xi}+y_0^{\eta}$.Therefore,

$$\mathrm{E}_g(\xi+\eta)=\mathrm{E}_g(\xi)+\mathrm{E}_g(\eta). \tag{8}$$

(Necessity)

$\forall \xi, \eta \in L^2(\Omega, \mathrm{F}_T, P), A \in \mathrm{F}_t$, we have

$$\mathrm{E}_g[(\xi+\eta)I_A] = \mathrm{E}_g[\mathrm{E}_g[\xi+\eta \mid \mathrm{F}_t]I_A]. \tag{9}$$

On the other hand,

$$\begin{aligned} &\mathrm{E}_g[(\xi+\eta)I_A] = \mathrm{E}_g(\xi I_A + \eta I_A) = \mathrm{E}_g(\xi I_A) + \mathrm{E}_g(\eta I_A) \\ &= \mathrm{E}_g[\mathrm{E}_g[\xi \mid \mathrm{F}_t]I_A] + \mathrm{E}_g[\mathrm{E}_g(\eta \mid \mathrm{F}_t I_A) \\ &= \mathrm{E}_g[(\mathrm{E}_g[\xi \mid \mathrm{F}_t] + \mathrm{E}_g[\eta \mid \mathrm{F}_t])I_A] \end{aligned} \tag{10}$$

Then, $\mathrm{E}_g[\xi+\eta \mid \mathrm{F}_t] = \mathrm{E}_g[\xi \mid \mathrm{F}_t] + \mathrm{E}_g[\eta \mid \mathrm{F}_t]$. Therefore, $y_t^{\xi+\eta} = y_t^{\xi} + y_t^{\eta}$.

Since

$$\begin{cases} y_t^{\xi+\eta} = \xi+\eta+\int_t^T g(s, y_s^{\xi+\eta}, z_s^{\xi+\eta})ds - \int_t^T z_s^{\xi+\eta}dW_s \\ y_t^{\xi} + y_t^{\eta} = \xi+\eta+\int_t^T g(s, y_s^{\xi}, z_s^{\xi}) + g(s, y_s^{\eta}, z_s^{\eta})ds - \int_t^T z_s^{\xi} + z_s^{\eta}dW_s \end{cases}, \tag{11}$$

Then, $z_t^{\xi+\eta} = \dfrac{dy_t^{\xi+\eta}.dW_t}{dt} = \dfrac{d(y_t^{\xi}+y_t^{\eta}).dW_t}{dt} = z_t^{\xi} + z_t^{\eta}$.

Therefore, $g(t, y_t^{\xi+\eta}, z_t^{\xi+\eta}) = g(t, y_t^{\xi}, z_t^{\xi}) + g(t, y_t^{\eta}, z_t^{\eta})$.

We are now in the way to prove that

$$g(t_0, a, b) + g(t_0, c, d) = g(t_0, a+c, b+d), \forall (t_0, a, b), (t_0, c, d) \in [0, T] \times R \times R^d. \tag{12}$$

Let

$$y_t^1 = \begin{cases} a - \int_{t_0}^t g(s, y_s^1, b)ds + \int_{t_0}^t bds, t_0 < t \le T \\ a, 0 \le t \le t_0 \end{cases}, \tag{13}$$

$$y_t^2 = \begin{cases} c - \int_{t_0}^t g(s, y_s^2, d)ds + \int_{t_0}^t dds, t_0 < t \le T \\ c, 0 \le t \le t_0 \end{cases}, \tag{14}$$

and $\xi = y_T^1$. Then， $\forall t \in (t_0, T]$， (y_t^1, b) satisfies

$$y_t^1 = \xi + \int_t^T g(s, y_s^1, b)ds - \int_t^T bdW_s.$$

Suppose that (y_t^{ξ}, z_t^{ξ}) is the solution of

$$y_t^{\xi} = \xi + \int_t^T g(s, y_s^{\xi})ds - \int_t^T z_s^{\xi}dW_s. \tag{15}$$

By the uniqueness solution of BSDE,

$$y_t^{\xi} = \begin{cases} a, 0 \le t \le t_0 \\ y_t^1, t_0 < t \le T \end{cases}, z_t^{\xi} = \begin{cases} 0, 0 \le t \le t_0 \\ b, t_0 < t \le T \end{cases}. \tag{16}$$

Especially, $(y_{t_0}^{\xi}, z_{t_0}^{\xi}) = (a, b)$. Similarly, we can find η such that $g(t_0, y_{t_0}^{\eta}, z_{t_0}^{\eta}) = g(t_0, c, d)$. From $g(t, y_t^{\xi+\eta}, z_t^{\xi+\eta}) = g(t, y_t^{\xi}, z_t^{\xi}) + g(t, y_t^{\eta}, z_t^{\eta})$, we get

$$g\left(t_0, a+c, b+d\right) = g\left(t_0, a, b\right) + g\left(t_0, c, d\right). \tag{17}$$

Since, $g(t, y, 0) = 0, \forall y_1, y_2$, then ,

$$g(t, y_1 + y_2, z) = g(t, y_1, z) + g(t, y_2, 0) = g(t, y_1, z). \tag{18}$$

In the same way, $g(t, y_1 + y_2, z) = g(t, y_2, z)$. Therefore, $g(t, y_1, z) = g(t, y_2, z), \forall y_1, y_2, t, z \in R$. That is $g(t, y, z)$ has nothing to do with y. In the following, we will prove that

$$g(t, lz) = lg(t, z), \forall t, l, z \in R. \tag{19}$$

Since

$$\begin{cases} g(t,0) = g(t,z) + g(t,-z) = 0 \\ g(t,z) = -g(t,-z) \end{cases}, \tag{20}$$

Then, $\forall m \in Z^+$, we have

$$\begin{cases} g(t, mz) = mg(t, z) \\ g(t, z) = g(t, m\dfrac{z}{m}) = mg(t, \dfrac{z}{m}). \\ g(t, \dfrac{z}{m}) = \dfrac{1}{m} g(t, z) \end{cases} \tag{21}$$

From Eq. 21, we get that for any rational number r, $g(t, rz) = rg(t, z)$.On the other hand, g is uniformly Lipschitz continuous in z, then $l \in R$, we have $g(t, lz) = lg(t, z)$.

Therefore, $g(t, z)$ is linear with z and $g(t, z) = h(t)z.$ Then $\forall \xi \in L^2(\Omega, \mathrm{F}_T, P), l \in R,$ we have

$$\mathrm{E}_g(l\xi) = l\mathrm{E}_g(\xi). \tag{22}$$

In summary, E_g is a linear expectation.

Theorem2. Under the assumptions of Theorem 1, there exists a probability measure Q which makes $\mathrm{E}_g(\xi) = E_Q(\xi)$.

Proof: $\forall \xi \in L^2(\Omega, \mathrm{F}_T, P)$,

$$y_t = \xi + \int_t^T g(s, z_s) ds - \int_t^T z_s dW_s = \xi + \int_t^T h(s) z_s ds - \int_t^T z_s dW_s, \tag{23}$$

Then, $-dy_t = h(t) z_t dt - z_t dW_t = -z_t(-h(t)dt + dW_t) = z(t) dW_t$.

From Girsanov Theorem, there exists a probability measure Q such that W_t is B.M under Q.

Hence, $y_t = \xi - \int_t^T z_t dW_t$ and $y_t = E_Q[\xi \mid \mathrm{F}_t], E_g(\xi) = y_0 = E_Q(\xi)$.

Conclusions

This paper has studied the sufficient and necessary condition for Peng's g-expectation to be linear.G-expectation is linear if and only if g has nothing to do with variable y, and g is linear in variable z.Moreover, there exists a probability measure Q, under which the linear expectation is equivalent to Peng's g-expectation.

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