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Characterizing Distributions by Linearity of Regression of Generalized Order Statistics

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Let $X_1, \ldots, X_n$ be a random sample from an absolutely continuous (with respect to Lebesgue measure) distribution with the corresponding generalized order statistics $X(1,n,\tilde{m},k), \ldots, X(n,n,\tilde{m},k)$. In this paper, we present some characterization of distributions when linearity of regression $E\left[X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)=x\right]=ax+b$ is identified.

Keywords: Conditional expectation; generalized order statistics; Linearity of regression.

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1. Introduction

Order statistics and record values are widely used in statistical modeling and inference; both models describe random variables arranged in order of magnitude. In the distribution theoretical sense, all of these models of ordered random variables are contained in the proposed concept of generalized order statistics. Let, for simplicity, $F(x)$ throughout denote an absolutely continuous distribution function with probability density function $f(x)$. The random variables $X(1,n,\tilde{m},k), \ldots, X(n,n,\tilde{m},k)$

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are called generalized order statistics based on $F(x)$, if their joint density function is given by

$$f_{X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)}(x_1,\dots,x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(1-F(x_i))^{m_i}f(x_i)\right) \times (1-F(x_n))^{k-1}f(x_n), \quad (1.1)$$

on the cone $F^{-1}(0) < x_1 \le \dots \le x_n < F^{-1}(1)$ of $\mathbb{R}^n$, with parameters $n \in \mathbb{N}$, $n \ge 2$, $k > 0$, $\tilde{m} = (m_1,\dots,m_{n-1}) \in \mathbb{R}^{n-1}$, $M_i = \sum_{j=i}^{n-1} m_j$, such that $\gamma_i = k+n-i+M_i > 0$ for all $i \in \{1,\dots,n-1\}$ (see, Kamps [9], [10]). Moreover, let $c_{r-1} = \prod_{j=1}^{r}\gamma_j$, $r = 1,\dots,n-1$, and $\gamma_n = k$.
Here, we will assume throughout that the parameters $\gamma_1,\dots,\gamma_n$ are pairwise different (see, Kamps and Cramer [11]) i.e.,

$$\gamma_i \ne \gamma_j, \quad i \ne j, \quad \text{for all } i,j \in \{1,\dots,n\}. \quad (1.2)$$

For $1 \le r < s \le n$, the marginal density function of $X(r,n,\tilde{m},k)$ and joint density function of $X(s,n,\tilde{m},k)$ and $X(r,n,\tilde{m},k)$ is respectively given by (see Kamps and Cramer, [11])

$$f_{X(r,n,\tilde{m},k)}(x) = c_{r-1}f(x)\sum_{i=1}^{r}a_i(r)(1-F(x))^{\gamma_i-1},$$

and

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x_r,x_s) = c_{s-1}\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(x_s)}{1-F(x_r)}\right)^{\gamma_i}\left[\sum_{i=1}^{r}a_i(r)(1-F(x_r))^{\gamma_i}\right] \times \frac{f(x_r)}{1-F(x_r)}\frac{f(x_s)}{1-F(x_s)}, \qquad x_r < x_s,$$

where $a_i(r) = \prod_{j=1,j\ne i}^{r}\frac{1}{\gamma_j-\gamma_i}$, $1 \le i \le r$ and $a_i^{(r)}(s) = \prod_{j=r+1,j\ne i}^{s}\frac{1}{\gamma_j-\gamma_i}$, $r+1 \le i \le s$. Also, the conditional distribution function of $X(s,n,\tilde{m},k)$ given $X(r,n,\tilde{m},k)$ is given by

$$f_{X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)}(x_s|x_r) = \frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{1-F(x_s)}{1-F(x_r)}\right)^{\gamma_i}\frac{f(x_s)}{1-F(x_s)}, \qquad x_r < x_s. \quad (1.3)$$

Now, this paper attempts to characterize some well-known continuous probability distributions based on equation

$$E[X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k) = x] = ax+b,$$

That illustrates linearity regression of a generalized order statistics on other one that is not necessarily adjacent. So many authors work on characterization of distribution based on linearity regression. For example Nagaraja [12] following Ferguson [7] for order statistics, Nagaraja [13], making use of some analogy between conditional distribution of record values and order statistics distribution and exploiting Ferguson [7] results. In Wesołowski and Ahsanullah [17] the first results on linearity regression non-adjacent order statistics appeared. In Dembińska and Wesołowski [5], [6]) the problem of linearity of regression for any possibly non-adjacent, order statistics and record values has been completely resolved under the mild and natural assumption of continuity of $F(x)$. Gupta and Ahsanullah [8] studied on the characterization results based on the conditional expectation of a function of non-adjacent order statistics (record value). Also Ahsanullah and Raqab [2], Raqab

and Abu-Lawi [14] and Samuel [16] characterized some distributions by conditional expectation of some functions of adjacent generalized order statistics.
In the next section, some characterizations of general forms of many well-known continuous probability distributions, which are generalization of the characterizations based on the order statistics and record values of Dembińska and Wesołowski [5], [6], are presented.

2. Characterizations by linearity of regression of $X(s,n,\tilde{m},k)$ on $X(r,n,\tilde{m},k)$

In this section, we are looking for characterization of exponential distribution, power distribution and Pareto distribution by linearity of regression of generalized order statistics. Before stating the main theorem of this section, we are introduced to describe these distributions.
The exponential distribution with λ and γ parameters defined by

$$F(x) = 1 - exp\{-\lambda(x-\gamma)\}; \quad x > \gamma,$$

where $\lambda > 0$ and γ are some real constants.
The power distribution with θ, μ and ν parameters defined by

$$F(x) = 1 - \left[\frac{\nu - x}{\nu - \mu}\right]^{\theta}; \quad \mu < x < \nu,$$

where $\theta > 0$ and $-\infty < \mu < \nu < +\infty$ are some constants.
Finally, the Pareto distribution with θ, μ and δ parameters defined by

$$F(x) = 1 - \left[\frac{\mu + \delta}{x + \delta}\right]^{\theta}; \quad x > \mu,$$

where $\theta > 0$ and μ, δ are some real constants such that $\mu + \delta > 0$.

Theorem 2.1. *Let $X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)$ be n generalized order statistics with absolutely continuous (with respect to Lebesgue measure) distribution function $F(x)$ and the corresponding probability density function $f(x)$ such that $E(|X(s,n,\tilde{m},k)|) < \infty$. if for $s > r$ and some real a and b*

$$E[X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k) = x] = ax + b, \tag{2.1}$$

then only the following three cases are possible:
1. $a = 1$, iff X be a random variable from an exponential distribution,
2. $a > 1$, iff X be a random variable from a Pareto distribution,
3. $a < 1$, iff X be a random variable from a power distribution.

Before we give the proof of the above result let us recall, following Rao and Shanbhag [15], an important result concerning possible solutions of an extended version of the integrated Cauchy functional equation. This theorem will be used later on in the course of the proof of Theorem 2.1.

Theorem 2.2. *Consider the integral equation:*

$$\int_{R_+} H(x+y)\mu(dy) = H(x) + c \quad \text{for [L] a.a. } x \in \mathbb{R}_+,$$

where c is a real constant, μ is a non-arithmetic σ-finite measure on $\mathbb{R}_+$ such that $\mu(\{0\}) < 1$ and $H : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a Borel measurable, either non-decreasing or non-increasing function that is *locally [L] integrable and is not identically equal to a constant [L] a.e. Then* $\exists\eta \in \mathbb{R}$ *such that*

$$\int_{\mathbb{R}_+} exp(\eta x)\mu(dx) = 1,$$

and H has the form

$$H(x) = \begin{cases} \gamma + \alpha(1 - exp(\eta x)) & \text{for [L]a.a. } x \quad \text{if } \eta \neq 0 \\ \gamma + \beta x & \text{for [L]a.a. } x \quad \text{if } \eta = 0, \end{cases} \tag{2.2}$$

where α*,* β *and* γ *are some constants. If* $c = 0$ *then* $\gamma = -\alpha$ *and* $\beta = 0$*. Now we are ready to prove our main result.*

Proof. Using (1.3) for the conditional probability distribution function such that $s > r$, we obtain

$$E[X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k) = x] = \int_x^\infty y \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \frac{f(x)}{\bar{F}(y)} dy,$$

where $\bar{F} = 1 - F$. Now, from (2.1), we get

$$\frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \int_x^\infty y \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i - 1} d\left(-\frac{\bar{F}(y)}{\bar{F}(x)}\right) = ax + b, \tag{2.3}$$

for F-almost all x's. Following the argument applied in Ferguson [7] it follows, that (l_F, r_F) is the support of the distribution defined by F and F is strictly increasing in this interval.
Substituting $t = \frac{\bar{F}(y)}{\bar{F}(x)}$, i.e. $y = \bar{F}^{-1}(t\bar{F}(x))$ (observe that $\bar{F}^{-1}$ exists because $\bar{F}$ is strictly decreasing in (l_F, r_F)) into (2.3), we have

$$\frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \int_0^1 \bar{F}^{-1}(tw) t^{\gamma_i - 1} dt = a\bar{F}^{-1}(w) + b,$$

Now substitute $\bar{F}(x) = w$, hence $x = \bar{F}^{-1}(w)$. Then

$$\frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \int_0^1 \bar{F}^{-1}(tw) t^{\gamma_i - 1} dt = a\bar{F}^{-1}(w) + b, \qquad 0 < w < 1,$$

Substitute once again $t = e^{-u}$ and $w = e^{\upsilon}$, also divide both sides by a. Then

$$\frac{c_{s-1}}{a\,c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \int_0^\infty \bar{F}^{-1}(e^{-(u+\upsilon)}) e^{-u\gamma_i} du = \bar{F}^{-1}(e^{-\upsilon}) + \frac{b}{a}, \qquad \upsilon > 0$$

Now let $H(\upsilon) = \bar{F}^{-1}(e^{-\upsilon})$ and from Theorem 2.2, we get

$$\mu(du) = \frac{c_{s-1}}{a\,c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) e^{-u\gamma_i} du,$$

and

$$\frac{c_{s-1}}{a\,c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \int_0^\infty e^{-x(\gamma_i - \eta)} dx = 1. \tag{2.4}$$

After substituting $t = e^{-x}$ in (2.4), we obtain

$$\frac{c_{s-1}}{a\,c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \int_0^1 t^{\gamma_i-\eta-1}\,dt = 1,$$

consequently

$$a = \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\gamma_i - \eta}. \tag{2.5}$$

On the other hand, we have

$$\int_0^\infty f_{s|r}(y|x)\,dy = 1,$$

then

$$\frac{c_{s-1}}{a\,c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s) \int_x^\infty \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i} \frac{f(y)}{\bar{F}(y)}\,dy = 1,$$

and hence

$$\frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\gamma_i} = 1,$$

consequently

$$\sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\gamma_i} = \frac{c_{r-1}}{c_{s-1}}. \tag{2.6}$$

From equations (2.5) and (2.6), we obviously observe that

1. $a = 1$ iff $\eta = 0$,
2. $a > 1$ iff $\eta > 0$,
3. $a < 1$ iff $\eta < 0$.

Consider now three possible cases:
1. If $\eta = 0$ and substitute (2.5) in (2.4), we obtain $a = 1$.
Thus from (2.2), we get

$$\bar{F}^{-1}(e^{-x}) = \gamma + \beta x.$$

Hence

$$\bar{F}(z) = e^{-\left(\frac{z-\gamma}{\beta}\right)} = e^{-\lambda(z-\gamma)}, \qquad z > \gamma,$$

where $\lambda = \frac{1}{\beta}$. Hence X be a random variable from an exponential distribution with λ, γ parameters. As regards we know (2.4) increasing in η, and

2. If $a > 1$ and $\eta > 0$, then

$$\bar{F}(z) = \left(\frac{-\alpha}{z-\alpha-\gamma}\right)^{1/\eta} = \left(\frac{\gamma-(\alpha+\gamma)}{z-(\alpha+\gamma)}\right)^{1/\eta} = \left[\frac{\mu+\delta}{z+\delta}\right]^{\theta}; \qquad z > \mu,$$

where $\delta = -(\alpha+\gamma)$, $\mu = \gamma$ and $\theta = \frac{1}{\eta} > 0$. Thus X be a random variable from a Parato distribution with θ, μ and δ parameters.
3. If $a < 1$ and $\eta < 0$, then

$$\bar{F}(z) = \left(\frac{\alpha+\gamma-z}{\alpha}\right)^{-1/\eta} = \left(\frac{\alpha+\gamma-z}{\alpha+\gamma-\gamma}\right)^{-1/\eta} = \left[\frac{\nu-z}{\nu-\mu}\right]^{\theta}; \qquad \mu < z < \nu,$$

where $\nu = \alpha+\gamma$, $\mu = \gamma$ and $\theta = -\frac{1}{\eta} > 0$. Thus X be a random variable from a power distribution with θ, μ and ν parameters.

□

Corollary 2.1. *In Theorem 2.1 if X be a random variables from an exponential distribution with γ and λ parameters, then*

$$a = 1,\ b = \frac{c_{s-1}}{\lambda\, c_{r-1}} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\gamma_i^2}.$$

Corollary 2.2. *In Theorem 2.1 if X be a random variable from a Pareto distribution with θ, μ and δ parameters, then*

$$a = \theta \frac{c_{s-1}}{\lambda\, c_{r-1}} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\theta\gamma_i - 1}, \qquad b = \delta \frac{c_{s-1}}{\lambda\, c_{r-1}} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{(\theta\gamma_i - 1)\gamma_i}.$$

Corollary 2.3. *In Theorem 2.1 if X be a random variables from a power distribution with θ, μ and ν parameters, then*

$$a = \theta \frac{c_{s-1}}{\lambda\, c_{r-1}} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{\theta\gamma_i + 1}, \qquad b = \nu \frac{c_{s-1}}{\lambda\, c_{r-1}} \sum_{i=r+1}^{s} \frac{a_i^{(r)}(s)}{(\theta\gamma_i + 1)\gamma_i}.$$

3. Application

Most of the know characterization results based on conditional expectation of order statistics and generalized order statistics can easily be deduced as special cases of Theorem 2.1.

Example 3.1. If we take $s = r+1$ and $m_1 = \cdots = m_{n-1} = m$ with $m = 0$ and $k = 1$, then using Theorem 2.1, we get the result of Ferguson [7] for order statistics.

Example 3.2. If we take $s = r+j$ and $m_1 = \cdots = m_{n-1} = m$ with $m = 0$ and $k = 1$, then using Theorem 2.1, we get the result of Dembińska and Wesołowski [5] for order statistics.

Example 3.3. If we take $s = r+1$, $r \geq 1$ and $m_1 = m_2 = \cdots = m_n = m = -1$, then we get the characterizations of distributions based on record values given by Nagaraja [13].

Example 3.4. If we take $s = r+2$, $r \geq 1$ and $m_1 = m_2 = \cdots = m_n = m = -1$, then we get the characterizations of distributions based record values given by Ahsanullah and Wesołowski [3].

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