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Testing EBU_{mgf} Class of Life Distributions based on Goodness of Fit approach

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Based on the goodness of fit approach, a new test is presented for testing exponentiality against"exponential Better than Used in moment generating function ordering class"(EBU_{mgf}). The critical values and the powers of this test are calculated. It is shown that the proposed test enjoys good power and performs better than previous tests in terms of power and Pitman's asymptotic efficiencies for several alternative. Finally sets of real data are used as examples to elucidate the use of the proposed test in practical application.

Keywords: Moment generating function, EBU, Hypothesis test, Pitman's efficiency, Goodness of fit.

2000 Mathematics Subject Classification: 60K10,62E10.

1. Introduction

The new better than used aging classes have been studied by reliability, statisticians, survival analysts and others. Such classes are defined by stochastic comparisons of the residual life of a used unit with the lifetime of new one. Such classes can be derived based on several notions of comparison between random variables. The stochastic and the increasing concave comparisons are used by Muller and Stoyan [1] and Shaked and Shanthikumar [2]. Formally if X and Y are two random variables then we say that X is smaller than Y in the:

(i) stochastic order (denoted by $X \leq_{st} Y$) if $E[\phi(X)] \leq E[\phi(Y)]$ for all increasing functions ϕ.
(ii) increasing concave order (denoted by $X \leq_{icv} Y$) if $E[\phi(X)] \leq E[\phi(Y)]$ for all increasing concave functions ϕ.

In the context of lifetime distributions the above ordering has been used to give characterizations and new definitions of aging classes. One of the most important approaches to the study of aging is based on the concept of additional residual life. Let X be a lifetime random variable such that distribution function F with $F(0) = 0$. Given a unit which has survived up to time t, its additional residual life (Barlow and Proschan,[3]) is given by

$$X_t = [X - t|X > t], t \in x : F(x) < 1$$

The comparison of the additional residual life at different times has been used to produce several notions of aging. In particular, X is said to be

(i) New better than used (denoted by $X \in NBU$) if $\bar{F}(x+t) \leq \bar{F}(t)\bar{F}(x)$ for all $t \geq 0$.
(ii) New better than used in the increasing concave order (denoted by $X \in NBU(2)$) if

$$\int_0^x \bar{F}(u+t)dx \leq \bar{F}(t)\int_0^x \bar{F}(u)du, \quad \forall x,t \geq 0$$

.

For more details about the former aging notions, one may refer to Bryson and Siddiqui [4] Barlow and Proschan [3] and Deshpande et la. [5]. Elbatal [6] compared the survival function of a component of age t to a new component having the exponential distribution as its survival function.

Definition 1.1. X is exponential better (worse) than used (denoted by $X \in EBU$) If

$$\bar{F}(x+t) \leq \bar{F}(t)e^{\frac{-x}{\mu}}, \quad \forall x,t \geq 0.$$

The closure properties under reliability operation, moment inequality, and heritage under shock model have been discussed by Elbatal [6]. Statisticians and reliability analysts studied exponential better than used classes of life distributions from various points of view. For more details we refer to Hendi et al. [7] for EBU class, Attia et al. [8] for EBUA class, Abdul moniem [9] for EBUASI class, Hendi and AL-Ghufily [10] for EBUC class and AL-Ghufily [11,12] for EBUCA class.

Given two non-negative random variables X and Y, with survival functions $\bar{F}$ and $\bar{G}$, respectively, X is said to be smaller than Y in the moment generating function ordering (denoted by $X \leq_{mgf} Y$) if and only if,

$$\int_0^\infty e^{\lambda x}\bar{F}(x)dx \leq \int_0^\infty e^{\lambda y}\bar{G}(y)dy \qquad \text{for all} \quad \lambda > 0.$$

Recently based on this notion, Abbas [13] introduced a new aging class of life distributions.

Definition 1.2. We say that X is exponential better than used in the moment generating function order (denoted by$X \in EBU_{mgf}$) if $X_t \leq_{mgf} Y \quad$ for all $\quad t > 0$, where Y is an exponential random variable with the same mean as X.

Equivalently, $X \in EBU_{mgf}$ if and only if,

$$\int_0^\infty e^{\lambda x}\bar{F}(x+t)dx \leq \frac{\mu}{1-\lambda\mu}\bar{F}(t), \quad \forall \lambda, t \geq 0. \tag{1.1}$$

Note that, the definition (1.2) is motivated by comparing the moment generating function of the life time X_t of a component of age t with the moment generating function of another new life time Y of a component which is distributed exponentially with mean μ.

In the current investigation, we present a procedure to test X is exponential versus it is EBU_{mgf} and not exponential in Section 2. In Section 3, the Pitman asymptotic efficiencies are calculated for some commonly used distributions in reliability. Monte Carlo null distribution critical points and the power estimates are simulated in Section 4. Finally numerical examples is presented in Section 5.

2. Hypothesis testing problem

Our goal in this section is to present a test statistic based on goodness of fit approach for testing H_0: F is exponential against H_1: F belongs to EBU_{mgf} class and not exponential. We propose the following measure of departure

$$\delta = \mu \int_0^\infty \bar{F}(t) dF_0(t) - (1 - \lambda\mu) \int_0^\infty \int_0^\infty e^{\lambda x} \bar{F}(x+t) dx dF_0(t),$$

where $F_0(t) = e^{-t}$

The following lemma is essential for the development of our test statistic.

Lemma 2.1. *If $\phi(\lambda) = \int_0^\infty e^{\lambda x} dF(x)$ and $\mu = \int_0^\infty \bar{F}(x)dx$, then*

$$\delta(\lambda) = \lambda(1+\mu)(1-\phi(-1)) - (\lambda\mu - 1)(1-\phi(\lambda))$$

Proof. Note that

$$\delta = \mu \int_0^\infty e^{-t} \bar{F}(t) dt - (1-\lambda\mu) \int_0^\infty \int_0^\infty e^{\lambda x - t} \bar{F}(x+t) dx dt$$
$$= \mu I_1 - (1-\lambda\mu) I_2.$$

One can show that

$$I_1 = \int_0^\infty e^{-t} \bar{F}(t) dt = E \int_0^X e^{-t} dt = 1 - \phi(-1),$$

and

$$I_2 = \int_0^\infty \int_0^\infty e^{\lambda x - t} \bar{F}(x+t) dx dt = \int_0^\infty \int_t^\infty e^{-t} e^{\lambda(u-t)} \bar{F}(u) du dt$$
$$= \frac{1}{\lambda(1+\lambda)} [\phi(\lambda) - 1 - \lambda(1 - \phi(-1))].$$

Thus the result follows. □

To make the test scale invariant, we let $\delta_1(\lambda) = \frac{\delta(\lambda)}{\mu^2}$.
Note that under $H_0 : \delta_1(\lambda) = 0$, while under $H_1 : \delta_1(\lambda) > 0$.
To estimate $\delta_1(\lambda)$, let $X_1, X_2, X_3, \cdots, X_n$ be a random sample from F, so the empirical form of $\delta_1(\lambda)$ is

$$\hat{\delta}_{1n}(\lambda) = \frac{1}{n^2 \bar{X}^2} \sum_{i=1}^n \sum_{j=1}^n [\lambda(1+X_i)(1-e^{-X_j}) - (\lambda X_i - 1)(1 - e^{\lambda X_j})]. \tag{2.1}$$

By defining

$$\phi(X_1, X_2) = \lambda(1+X_1)(1-e^{-X_2}) - (\lambda X_1 - 1)(1-e^{\lambda X_2}),$$

and define the symmetric kernel

$$\psi(X_1, X_2) = \frac{1}{2!} \sum_R \phi(X_i, X_j),$$

where the sum is over all arrangements of X_i and X_j, this leads to $\hat{\delta}_{1n}(\lambda)$ is equivalent to U- statistic

$$U_n = \frac{1}{\binom{n}{2}} \sum_R \phi(X_i\,, X_j).$$

The next result summarizes the asymptotic normality of $\hat{\delta}_{1n}(\lambda)$.

Theorem 2.1. *As $n \to \infty$, $\sqrt{n}(\hat{\delta}_{1n}(\lambda) - \delta_1(\lambda))$ is asymptotically normal with mean 0 and variance is σ^2 given in* (2.5)*. Under H_0 , the variance is reduced to* (2.6)*.*

Proof. Let

$$\begin{aligned} \eta_1(X_1) &= E[\phi(X_1\,, X_2)\,|X_1] \\ &= \frac{\lambda}{2}(1+X_1) + \frac{\lambda}{1-\lambda}(\lambda X_1 - 1), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \eta_2(X_2) &= E[\phi(X_1\,, X_2)\,|X_2] \\ &= 2\lambda(1-e^{-X_2}) - (\lambda-1)(1-e^{\lambda X_2}). \end{aligned} \tag{2.3}$$

Considering $\eta(X) = \eta_1(X_1) + \eta_2(X_2)$, gives

$$\eta(X) = \left\{ \frac{\lambda(1+\lambda)}{2(1-\lambda)} X - 2\lambda e^{-X} - (1-\lambda)e^{\lambda X} - \frac{\lambda(1+\lambda)}{2(1-\lambda)} + \lambda + 1 \right\}. \tag{2.4}$$

In view of (2.4), the variance is

$$\sigma^2 = Var\left\{ \frac{\lambda(1+\lambda)}{2(1-\lambda)} X - 2\lambda e^{-X} - (1-\lambda)e^{\lambda X} \right\}. \tag{2.5}$$

Under H_0 it is easy to prove that $\mu_0 = E[\eta(X)] = 0$ and the variance σ_0^2 reduces to

$$\sigma_0^2 = \frac{\lambda^2(1+\lambda)^2(2\lambda^2 - \lambda + 2)}{12(1-\lambda)^2(2-\lambda)(1-2\lambda)}. \tag{2.6}$$

□

3. The Pitman Asymptotic Efficiency (PAE)

To asses the quality of this procedures, we evaluate its Pitman asymptotic efficiency (PAE) for some commonly used distributions in reliability, these are:

(1) Linear failure rate family (LFR): $\bar{F}_\theta(x) = \exp(-x - \frac{\theta}{2}x^2)\,, x > 0\,, \theta \geq 0$.
(2) Makeham family:$\bar{F}_\theta(x) = \exp(-x + \theta(x + e^{-x} - 1))\,, x > 0\,, \theta \geq 0$.
(3) Weibull family: $\bar{F}_\theta(x) = \exp(-x^\theta)\,, x > 0\,, \theta > 0$.

The PAE is defined by

$$PAE(\delta) = \frac{1}{\sigma_0} \left| \frac{d\delta}{d\theta} \right|_{\theta \to \theta_0},$$

where

$$\delta_\theta = \lambda(1+\mu_\theta)(1-\phi_\theta(-1)) - (\lambda\mu_\theta - 1)(1-\phi_\theta(\lambda)).$$

The PAE($\hat{\delta}$) can be written as,

$$\text{PAE}(\hat{\delta}, F) = \frac{1}{\sigma_0} \left| \lambda\mu'_\theta(\phi_\theta(\lambda) - \phi_\theta(-1)) - \lambda(1+\mu_\theta)\phi'_\theta(-1) + (\lambda\mu_\theta - 1)\phi'_\theta(\lambda) \right|,$$

where $\phi'_\theta(\lambda) = \int_0^\infty e^{\lambda x} dF'_\theta(x)$ and $\mu'_\theta = \int_0^\infty \bar{F}'_\theta(x)dx$.
After some mathematical calculations we get

$$\text{PAE}(\hat{\delta}, LFR) = \frac{1}{\sigma_0} \left| \frac{\lambda(1+\lambda)^2}{4(1-\lambda)^2} \right|,$$

$$\text{PAE}(\hat{\delta}, Makeham) = \frac{1}{\sigma_0} \left| \frac{\lambda(1+\lambda)(2+\lambda)}{12(1-\lambda)(2-\lambda)} \right|$$

and

$$\text{PAE}(\hat{\delta}, Weibull) = \frac{1}{\sigma_0} \left| \begin{array}{l} 2\lambda \int_0^\infty (x-1)e^{-2x} lnx dx + (1-\lambda) \int_0^\infty (x-1)e^{-x(1-\lambda)} lnx dx \\ -\frac{\lambda(1+\lambda)}{2(1-\lambda)} \int_0^\infty xe^{-x} lnx dx - (1+\lambda) \end{array} \right|.$$

Table 1. Pitman asymptotic efficiencies for various values of λ

	$\hat{\delta}_n$		U_n	δ_3	$\delta^{(2)}_{F_n}$
	λ				
LFR	0.01	0.8746	0.433	0.408	0.217
	0.02	0.8829			
	0.03	0.89117			
	0.21	0.98619			
	0.22	0.98717			
	0.23	0.9875			
Makeham	0.01	0.28863	0.144	0.039	0.144
	0.02	0.2885			
	0.03	0.2870			
	0.21	0.2649			
	0.22	0.2623			
	0.23	0.25962			
Weibull	0.01	1.1990	0.132	0.170	0.05
	0.02	1.1972			
	0.03	1.19496			
	0.21	1.08451			
	0.22	1.07354			
	0.23	1.06193			

Table 2. Critical values of statistic $\hat{\hat{\delta}}_{1n}(\lambda)$ at $\lambda = 0.23$

n	0.01	0.05	0.10	0.90	0.95	0.99
2	-0.020	-0.007	0.002	0.108	0.118	0.129
3	-0.088	-0.048	-0.021	0.088	0.098	0.113
4	-0.106	-0.046	-0.025	0.077	0.086	0.101
5	-0.101	-0.050	-0.025	0.069	0.077	0.089
6	-0.124	-0.052	-0.030	0.064	0.073	0.089
7	-0.112	-0.052	-0.030	0.060	0.070	0.079
8	-0.102	-0.051	-0.030	0.057	0.065	0.079
9	-0.108	-0.050	-0.028	0.054	0.062	0.080
10	-0.107	-0.048	-0.028	0.052	0.060	0.073
11	-0.094	-0.052	-0.030	0.049	0.057	0.071
12	-0.096	-0.044	-0.025	0.048	0.053	0.065
13	-0.096	-0.048	-0.029	0.046	0.052	0.066
14	-0.078	-0.043	-0.027	0.044	0.051	0.063
15	-0.095	-0.049	-0.028	0.043	0.049	0.059
16	-0.093	-0.045	-0.030	0.042	0.048	0.060
17	-0.094	-0.043	-0.027	0.041	0.047	0.058
18	-0.081	-0.044	-0.027	0.039	0.045	0.054
19	-0.076	-0.040	-0.029	0.039	0.045	0.054
20	-0.073	-0.041	-0.027	0.039	0.045	0.053
21	-0.084	-0.041	-0.028	0.037	0.043	0.054
22	-0.072	-0.040	-0.026	0.036	0.042	0.054
23	-0.078	-0.041	-0.028	0.035	0.041	0.051
24	-0.074	-0.042	-0.026	0.035	0.040	0.050
25	-0.071	-0.044	-0.029	0.035	0.040	0.049
26	-0.066	-0.039	-0.027	0.033	0.040	0.049
27	-0.067	-0.040	-0.027	0.033	0.041	0.048
28	-0.058	-0.039	-0.027	0.032	0.038	0.048
29	-0.080	-0.039	-0.026	0.032	0.039	0.047
30	-0.068	-0.039	-0.026	0.031	0.036	0.046
31	-0.059	-0.036	-0.025	0.031	0.036	0.046
32	-0.062	-0.037	-0.027	0.031	0.035	0.047
33	-0.058	-0.041	-0.026	0.030	0.036	0.045
34	-0.056	-0.036	-0.025	0.030	0.036	0.044
35	-0.056	-0.036	-0.023	0.030	0.035	0.043
36	-0.054	-0.034	-0.023	0.028	0.033	0.043
37	-0.052	-0.034	-0.024	0.028	0.033	0.041
38	-0.054	-0.034	-0.024	0.028	0.033	0.042
39	-0.055	-0.033	-0.022	0.028	0.033	0.043
40	-0.055	-0.033	-0.023	0.028	0.032	0.041
41	-0.051	-0.033	-0.023	0.028	0.034	0.043
42	-0.056	-0.033	-0.021	0.028	0.033	0.041
43	-0.055	-0.032	-0.023	0.027	0.033	0.038
44	-0.059	-0.031	-0.022	0.027	0.031	0.040
45	-0.049	-0.031	-0.023	0.026	0.031	0.040
46	-0.058	-0.031	-0.021	0.026	0.030	0.040
47	-0.049	-0.030	-0.020	0.026	0.030	0.039
48	-0.055	-0.030	-0.021	0.027	0.031	0.041
49	-0.049	-0.031	-0.020	0.025	0.029	0.037
50	-0.054	-0.030	-0.023	0.025	0.030	0.038

Table 1 gives the efficiencies of our proposed test $\hat{\delta}_{1n}(\lambda)$ for various values of λ comparing with the tests given by Kango [14](U_n), Mugdadi and Ahmad [15](δ_3)and Mahmoud and Abdul Alim [16]($\delta_{F_n}^{(2)}$).

One can note that our test is more efficient for all used alternatives.

4. Monte Carlo Null Distribution Critical Points

In practice, simulated percentiles are commonly used by applied statisticians and reliability analyst. Next, we simulate the Monte Carlo null distribution critical points for $\hat{\delta}_{1n}(\lambda)$ in (2.1) based on 10000 simulated sample 2(1)50 from the standard exponential distributions. Table 2 gives these percentile points of the statistics $\hat{\delta}_{1n}(\lambda)$ at $\lambda = 0.23$.

In view of Table 2, it is noticed that the critical values are increasing as the confidence level increasing and is almost decreasing as the sample size increasing.

4.1. *The Power Estimates*

Table 3 shows the power estimate of the test statistic $\hat{\delta}_{1n}(\lambda)$ at the significant level 0.05 using LFR, Makeham and Weibull distributions. The estimates are based on 10000 simulated samples for sizes $n = 10, 20$ and 30.

Table 3. Power estimates at $\lambda = 0.23$

	n	$\theta = 2$	$\theta = 3$	$\theta = 4$
LFR	10	0.404	0.504	0.588
	20	0.633	0.740	0.806
	30	0.804	0.886	0.924
Makeham	10	0.914	0.908	0.918
	20	0.992	0.992	0.986
	30	1.000	1.000	1.000
Weibull	10	0.780	0.994	1.000
	20	0.980	1.000	1.000
	30	0.999	1.000	1.000

It is clear from Table 3 that our test has good powers for all alternatives and the power estimate increase as the the sample size increases. The powers are getting as greater as the class departs the exponential distribution.

5. Numerical Examples

Example 5.1. The following data represent 39 liver cancers patients taken from Elminia cancer center Ministry of Health - Egypt, which entered in (1999)(see Attia et al. [8]). The ordered life times (in years) are:

0.027	0.038	0.038	0.038	0.038	0.038	0.041	0.047	0.049	0.055
0.055	0.055	0.055	0.055	0.063	0.063	0.066	0.071	0.082	0.082
0.085	0.110	0.314	0.140	0.143	0.164	0.167	0.184	0.195	0.203
0.206	0.238	0.263	0.288	0.293	0.293	0.293	0.318	0.411	

It was found that $\hat{\delta}_{1n} = 0.0523$ and this value exceeds the tabulated critical value in Table 2. It is evident that at the significant level 0.05 this data set has EBU_{mgf} property.

Example 5.2. Consider the following data, which represent failure times in hours, for a specific type of electrical insulation in an experiment in which the insulation was subjected to a continuously increasing voltage stress (Lawless [17], p.138):

0.205	0.363	0.407	0.477	0.72	0.782
1.178	1.255	1.592	1.635	2.31	

It was found that $\hat{\delta}_{1n} = 0.0109$ which is less than the tabulated critical value in Table 2. Then we accept H_0 which states that the data set has exponential property.

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